The breaking of quantum double symmetries by defect condensation

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Abstract

In this paper, we study the phenomenon of Hopf or more specifically quantum double symmetry breaking. We devise a criterion for this type of symmetry breaking which is more general than the one originally proposed in [1,2], and therefore extends the number of possible breaking patterns that can be described consistently. We start by recalling why the extended symmetry notion of quantum double algebras is an optimal tool when analyzing a wide variety of two dimensional physical systems including quantum fluids, crystals and liquid crystals. The power of this approach stems from the fact that one may characterize both ordinary and topological modes as representations of a single (generally non-Abelian) Hopf symmetry. In principle a full classification of defect mediated as well as ordinary symmetry breaking patterns and subsequent confinement phenomena can be given. The formalism applies equally well to systems exhibiting global, local, internal and/or external (i.e. spatial) symmetries. The subtle differences in interpretation for the various situations are pointed out. We show that the Hopf symmetry breaking formalism reproduces the known results for ordinary (electric) condensates, and we derive formulae for defect (magnetic) condensates which also involve the phenomenon of symmetry restoration. These results are applied in two papers which will be published in parallel [3,4].

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1 Introduction and motivations

In recent years it has become clear that in the setting of two-dimensional (quantum) physics, the notion of quantum double algebras or quantum groups plays an important role. One of the main reasons is that these extended symmetry concepts allow for the treatment of topological and ordinary quantum numbers on equal footing. This means that the representation theory of the underlying - hidden - Hopf algebra $\mathcal{A}$ labels both the topological defects and the ordinary excitations. This is also the case in non-Abelian situations where the mutual dependence of these dual quantum numbers would otherwise be untractable. Moreover, as the Hopf algebras involved turn out to be quasi-triangular they are naturally endowed with a R-matrix which describes the topological interactions between the various excitations in the medium of interest [5,6]. This theory has found interesting applications in the domain of Quantum Hall liquids [7,8,9,10,11], exotic phases in crystals and liquid crystals [3], as well as 2-dimensional gravity [12]. It also appears to furnish the appropriate language in the field of topological quantum computation [13,14,15,16].

Once this (hidden) extended symmetry was identified, the authors of [1,2] studied the breaking the Hopf symmetry by assuming the formation of condensates, respectively of ordinary (which we call *electric*), defect (*magnetic*), or mixed (*dyonic*) type. As was to be expected, the ordinary condensates reproduce the conventional theory of symmetry breaking, though the analysis of confinement of topological degrees of freedom, using the braid group, is not standard. In [1,2] it was shown that when considering Hopf symmetry breaking, the usual formalism of symmetry breaking had to be extended with significant novel ingredients. One assumes the condensate to be represented by a fixed vector in some nontrivial representation of the Hopf algebra $\mathcal{A}$. This leads to the definition of an intermediate algebra $\mathcal{T}$ as the suitably defined stabilizer subalgebra of the condensate. The complication that arises at this level is that certain representations of $\mathcal{T}$ may braid nontrivially with the condensate, which in turn means that the condensate cannot be single valued around a particle belonging to such a representation. If this happens to be the case, it implies that such particles (representations) necessarily are confined. The effective low energy theory of the non-confined degrees of freedom is then characterized by yet a smaller algebra called $\mathcal{U}$. So the breaking of Hopf symmetries involves three algebras: the unbroken algebra $\mathcal{A}$, the intermediate algebra $\mathcal{T}$, and the unconfined algebra $\mathcal{U}$.

In this paper we show that the assumptions made about the structure of the intermediate residual symmetry algebra $\mathcal{T}$ in [1,2] can be relaxed. This leads to a new definition of the residual symmetry algebra $\mathcal{T}_r$ which contains $\mathcal{T}$ as a subalgebra, which means that the residual phase may have a richer spectrum. We will discuss the new criterion in some detail and point out its importance.

We conclude by deriving general formulae for $\mathcal{T}_r$ and $\mathcal{U}$ for the case of electric and defect condensates, in phases whose corresponding Hopf algebra is what we call a *modified quantum double* $F(H_m) \times \mathbb{C}H_{el}$, where H_m and H_{el} are finite groups. H_m is the *magnetic group*, i.e. the defect group. H_{el} is the *electric group*, or the residual symmetry group[1]. We know what to expect for electric condensates, and for that case our method reproduces the known results. The problem of defect condensates is more interesting, and provided the main motivation for this work. In this paper we focus on the basic structure and the more formal aspects of the symmetry breaking analysis and we mainly present general results, involving both confinement and liberation phenomena. In two separate papers we give detailed applications of these results: one paper on defect mediated melting [4] (which discusses also the phenomenon of symmetry restoration), and another on the classification of defect condensates in non-abelian nematic crystals [3].

The starting point in this paper will be situations where a continuous internal or external (space) symmetry G is broken to a possibly non-abelian, discrete subgroup H. This may happen through some Higgs mechanism in which case we speak of *discrete gauge theories* [17], or by forming some liquid crystal for example. Topological (line) defects can then be labelled by the elements h of the discrete residual symmetry group[2] H. Familiar examples of such defects are dislocations, disclinations, vortices, flux-tubes etc. Such a theory has a (hidden) Hopf symmetry corresponding to the so called *quantum double* $D(H)$ of H. This Hopf symmetry is an extension of the usual group symmetry[3]. The elements a of $D(H)$ are denoted by $a = (f \times h)$, where f is some function on the group H, and h some element of the group algebra $\mathbb{C}H$ of H (later on we will omit the $\times$ sign). A basis for the space of functions $F(H)$ on a discrete group H is a set of delta functions which project on each group element. We denote these projectors by P_g, these are just delta functions δ_g ($g \in H$) defined by: $\{P_g(g') \equiv \delta_g(g') = \delta_{g,g'} \mid \forall g' \in H \}$. The element f is basically a measurement operator (projecting on a subspace), while h involves a symmetry operation. In other words, if we denote the elements of the group H by g_i, where i labels the group elements, we can write

$$f = \sum_i \lambda_i P_{g_i}$$
$$h = \sum_i \mu_i g_i,$$

where the λ_i and μ_i are complex numbers. This follows from the fact that the P_{g_i} span $F(H)$, and the g_i span the group algebra of H, which we denote by $\mathbb{C}H$. An alternate notation for $D(H)$ is $F(H) \times \mathbb{C}H$, which shows that it is a combination of $F(H)$ and $\mathbb{C}H$.

The multiplication of two elements of $D(H)$ is defined by

$$(f_1 \times h_1)(f_2 \times h_2)(x) = f_1(x) f_2(h_1^{-1} x h_1) \otimes h_1 h_2 \quad x \in H \tag{1}$$

Note that the multiplication on the group part is just the ordinary group multiplication, but the multiplication of the functions is not just pointwise but twisted by a conjugation. This is a nontrivial feature which implies that the product is not a simple tensor product. There are two structures on a Hopf algebra $\mathcal{A}$ of special interest to our purposes (for an introduction to Hopf algebras, see for example [18]). One is the so called *counit* ε which turns out to be of particular relevance in the present context. ε is an algebra morphism from the Hopf algebra to $\mathbb{C}$, defined through $\varepsilon(f \times h) = f(e)$. The other structure we wish to mention is the *antipode* S which is a kind of inverse needed to introduce conjugate or antiparticle representations. S is an algebra antimorphism from $\mathcal{A}$ to $\mathcal{A}$, which satisfies $S(P_g \times h) \equiv P_{h^{-1}g^{-1}h} \times h^{-1}$. A quantum double is further endowed with an invertible element $R \in \mathcal{A} \otimes \mathcal{A}$ which implements the braiding of representations. It encodes the topological interactions and in particular the possible exotic "quantum statistics properties" of the excitations in the system.

The sectors of the theory, or the physical excitations for that matter, can be labelled by the irreducible representations of $D(H)$. These are denoted as Π_α^A, where the label A denotes the conjugacy class $C_A \subset H$ to which the topological (magnetic) charge of the excitation belongs, whereas α denotes a representation of the centralizer group N_A of a chosen preferred element h_A of C_A. α fixes the "ordinary" or electric charge. Clearly, Π_α^A describes in general a mixed electric and magnetic, usually called *dyonic*, excitation. Pure defects or ordinary excitations correspond to the special cases where one of the two labels becomes trivial. We denote the carrier space on which Π_α^A acts as V_α^A. The Hopf algebra structure requires a well defined comultiplication which ensures the existence of well defined fusion or tensor product rules for the representations of the algebra.

The quantum double appears when the defect group H_m, which we call the magnetic group, and the residual symmetry group H_{el}, which we call the electric group, are the same group H. This is often the case, because the magnetic group is equal to $H_m = \Pi_1(G/H_{el}) = H_{el}$, when G is connected and simply connected. However, there are cases where H_m and H_{el} are not equal. For example, if we want to distinguish between chiral and achiral phases, we have to include inversions (or reflections). The inclusion of inversions does not alter the defect group (because the inversions live in a part of G that isn't connected to the identity), but it can alter the electric group. We are thus led to the structure of a modified quantum double $F(H_m) \times \mathbb{C}H_{el}$, which is a special case of what is called a bicrossproduct[4] of the Hopf algebras $F(H_m)$ and $\mathbb{C}H_{el}$. The structures in $F(H_m) \times \mathbb{C}H_{el}$ are very similar to those in $D(H)$,

we give them in appendix A. The crucial point is that the electric group acts on the magnetic group, namely a residual symmetry transformation $h \in H_{el}$ transforms the defects. We denote the action of $h \in H_{el}$ on $g \in H_m$ by $h \cdot g$. In the case of $D(H)$, the action is simply conjugation: $h \cdot g = hgh^{-1}$. To obtain the structures of $D(H)$ from those of $F(H_m) \times \mathbb{C}H_{el}$, one basically replaces all occurrences of $h \cdot g$ by hgh^{-1}.

2 Breaking, braiding and confinement

In the previous section we mentioned that a Hopf symmetry captures the fusion and braiding properties of excitations of spontaneously broken phases. In particular we mentioned the modified quantum double $F(H_m) \times \mathbb{C}H_{el}$, which is relevant for a phase with magnetic group H_m and electric group H_{el}. In this section we will show that this Hopf symmetry description allows for a systematic investigation of phase transitions induced by the condensation of some excitation. Since electric, magnetic and dyonic modes are treated on equal footing, we will unify the study of phase transitions induced by the formation of condensates of all three types of modes.

The use of Hopf symmetries to analyze phase transitions was pioneered by Bais, Schroers and Slingerland [1,2]. In this article we stay close to their work. An important departure however, is the definition of "residual symmetry operators". Based on physical arguments we put forward a new definition, which, though similar, is more general than the previous one.

2.1 Breaking

We want to study the ordered phases arising if a condensate forms, corresponding to a non-vanishing vacuum expectation value of some vector which we denote as $|\phi_0>$ in some representation of the Hopf algebra. This vacuum vector breaks the Hopf-symmetry $\mathcal{A}$, and just as in the conventional cases of spontaneous symmetry breaking we have to analyze what the residual symmetry algebra $\mathcal{T}$ is.

The natural criterion to determine the residual symmetry that was proposed in [1,2] is

$$\Pi_\phi(a) \mid \phi_0\rangle = \varepsilon(a) \mid \phi_0\rangle \qquad \forall \quad a \in \mathcal{T}, \tag{2}$$

where Π_ϕ is the representation in which $\mid \phi_0\rangle$ lives. The motivation for this criterion is that an operator a is a residual symmetry operator if it acts on

$|\ \phi_0\rangle$ as the trivial representation ε.

This condition was subsequently analyzed in [1,2], where it was argued that one must impose further restrictions on $\mathcal{T}$. Namely, it was imposed that $\mathcal{T}$ be the *maximal Hopf subalgebra* of $\mathcal{A}$ that satisfies the condition (2). This restriction was made under the quite natural physical assumption that one should be able to fuse representations of $\mathcal{T}$.

We have taken a closer look at this restriction, and the point we make in this paper is that $\mathcal{T}$ need *not* be a Hopf algebra. It turns out that there is no physical reason to require $\mathcal{T}$ to be Hopf, and in fact lifting that requirement opens up a number of very interesting new possibilities that appear to be essential in describing realistic physical situations. Rather than talking about a single Hopf-algebra $\mathcal{T}$ we have to distinguish two algebras (not necessarily Hopf) $\mathcal{T}_r$ and $\mathcal{T}_l$. One can choose either one and we choose to define the breaking with $\mathcal{T}_r$. $\mathcal{T}_r$ is defined as the set of operators in $\mathcal{A}$ that satisfy

$$(1 \otimes \Pi_\phi)\Delta(a)(1 \otimes |\phi_0>) = a \otimes |\phi_0> . \tag{3}$$

We will explain the physical motivation for this criterion later on. Stated in words: the residual symmetry operators belonging to $\mathcal{T}_r$ are operators that cannot distinguish whether a given particle has fused with the condensate $|\phi_0>$ or not. Therefore these operators are so to say "blind" to the condensate, and the measurements they are related to are not affected by the presence of the condensate. We note for now that $\mathcal{T}_r$ need not be a Hopf algebra, but that it contains the Hopf algebra $\mathcal{T}$ as a subalgebra. Still, all its elements a satisfy $\Pi_\phi(a)|\phi_0>= \varepsilon(a)|\phi_0>$, thus the operators in $\mathcal{T}_r$ also act as the vacuum representation on $|\phi_0>$.

The important difference with the previous analysis of [1,2], is that, as we have lifted the Hopf algebra requirement on $\mathcal{T}$, we will in general no longer have a well defined tensor product for the representations of $\mathcal{T}_r$. At first sight this seems problematic from a physical point of view, but the opposite turns out to be the case, the ambiguity that arises reflects the physical situation perfectly well. Imagine that we bring some localized excitation into the medium, moving it in from the left, then it may well be that the vacuum is not single valued when "moved" around this excitation. If such is the case the excitation will be *confined*, and we obtain a situation as depicted in Figure 1. The confined particle is connected to a domain wall. The condensate to the right of the wall is in the state $|\phi_r>$, and to the left it is in the state $|\phi_l>$. Thus the question arises where the condensate takes on the original value $|\phi_0>$. When we say the condensate is in the state $|\phi_0>$, and we want to study confined excitations, we must specify whether the state of the condensate is $|\phi_0>$ to the far left or the far right of the plane, away from any excitations. We cannot a priori impose that the state of the condensate to the far left and the far

right is the same, because then we wouldn't allow for confined excitations. Choosing to set $|\phi_r >= |\phi_0 >$, i.e. setting the state of the condensate to the far right, gives $\mathcal{T}_r$ as intermediate symmetry algebra. Choosing $|\phi_l >= |\phi_0 >$ gives us a different intermediate algebra $\mathcal{T}_l$. As mentioned before we'll work with $\mathcal{T}_r$.

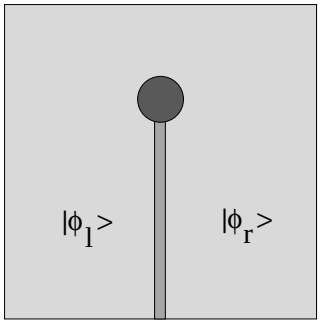

Fig. 1. A confined particle in the broken vacuum with a domain wall attached. The state $|\phi_r >$ of the vacuum on the right differs from the state $|\phi_l >$ of the vacuum on the left.

One may now restrict oneself to the unconfined representations and obtain that they form the representations of yet another algebra, the unconfined (Hopf) algebra $\mathcal{U}$. Putting it all together we have arrived at the picture in figure 2 for the generic symmetry breaking scheme. The mathematical structure is appealing and exactly reflects the subtlety of having to deal with confined particle representations.
One may show that the maximal Hopf-subalgebra $\mathcal{T}$ satisfying the criterion (2) is contained in the intersection of $\mathcal{T}_r$ and $\mathcal{T}_l$:

$$\mathcal{T} \subseteq \mathcal{T}_r \cap \mathcal{T}_l. \tag{4}$$

It is also possible to demonstrate that *if* $\mathcal{T}_r$ is a Hopf algebra then we have necessarily the relation $\mathcal{T} = \mathcal{T}_r = \mathcal{T}_l$. We will show these and other results later.

In the following sections we motivate the Hopf symmetry breaking formalism by way of examples, and we fully analyze electric and defect condensates starting with phases whose Hopf symmetry is a modified quantum double $F(H_m) \times \mathbb{C}H_{el}$. The breaking by electric condensates proceeds along familiar lines (i.e. Landau's theory of phase transitions). However, our arguments are more complete also in these situations, because within our formalism we arrive at a residual symmetry algebra $\mathcal{T}_r$ where all defects are still present, and we then find $\mathcal{U}$ by analyzing which defects are confined, and removing them from the spectrum.

The fact that we reproduce the theory of electric condensates is encouraging. More interesting nontrivial results occur when we analyze defect condensates.

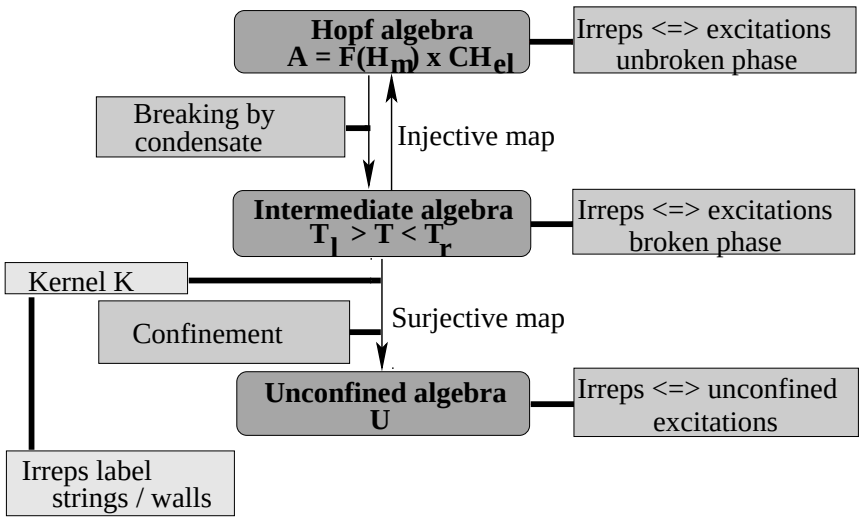

Fig. 2. A schematic of the quantum symmetry breaking formalism. One has to distinguish three levels: The unbroken Hopf algebra $D(H)$, the intermediate algebra $\mathcal{T}_r$, and the effective unconfined algebra $\mathcal{U}$. This scheme should be compared with the simple scheme of ordinary symmetry breaking where one only has to distinguish between the unbroken group G and some residual symmetry group $H \subset G$.

Applications of the results of this paper to defect condensates in general classes of nematic liquid crystals [3], and defect mediated melting [4], will be published elsewhere.

Excitations in the broken phase, which has $\mathcal{T}_r$ as its symmetry algebra, should form representations of $\mathcal{T}_r$. We must now take a close look at these excitations: some of them turn out to be attached to a domain wall, i.e. their presence would require a half-line singularity in the condensate, and such a wall costs a finite amount of energy per unit length (as we will show later in a simple example). To see this, we must first discuss braiding. It is also useful to discuss the issue of braiding, to point out characteristic differences between phases in which local and global symmetries are broken.

2.2 Braiding

Braiding addresses the question of what happens to a multi-particle state when one particle is adiabatically transported around another. To answer this question one is led to introducing a braid operator, whose properties we briefly recall. We focus our discussion on phases with a $F(H_m)\times\mathbb{C}H_{el}$ Hopf symmetry. We start with some heuristic arguments first involving the braiding of a defect and an ordinary excitation, then the braiding of two defects, and finally that of two ordinary excitations. We will assume throughout that $H_{el} = H_m = H$ is discrete.

Consider the braiding of an electric mode with a defect in the gauge theory case, because the global case is more subtle. This question goes back to the very definition (or measurement) of the defect [19]. If we have a discrete gauge theory with a defect g, then g is the "topological charge" of the defect, and it is defined as the path-ordered exponential of the gauge field along a path around the defect:

$$g = Pe^{i \oint \vec{A} \cdot d\vec{x}}.$$

This phase factor basically corresponds to the change of the wave function $|v>$ (which takes a value in some irreducible representation of the gauge group H) of some particle coupled to the gauge field when it is parallel transported around the defect. The outcome is

$$|v> \mapsto |v'> = Pe^{i \oint \vec{A} \cdot d\vec{x}} |v> = g \cdot |v> . \tag{5}$$

So $|v'>$ is defined through the condition of a vanishing covariant derivative: $D_i|v'> = 0$.

Thus *as an electric mode is transported around the defect, it is acted on by the topological charge g defining the defect*. We can choose a gauge such that the field v is constant everywhere except along a line going "up" from the origin. Then half-braiding already gives $v \mapsto g \cdot v$, because underneath the defect v is constant, i.e. v braids trivially with the defect if it passes below the defect. This just shows that this distinction is gauge dependent, because the operator itself is. Indeed, even the path ordered exponential in (5) is not gauge invariant in the nonabelian case, however the conjugacy class to which it belongs is. In a locally invariant theory any physical outcome can only depend on some gauge invariant expression involving this path ordered exponential.

Let us now discuss the braiding of two defects [17,6]. If we carry a defect h counterclockwise around a defect g then h gets conjugated by g and becomes ghg^{-1}, while g gets conjugated to hgh^{-1}. We can encode this behavior by defining a *braid operator* $\mathcal{R}$. If $|g>$ lives in V^A and $|h>$ in V^B, then $\mathcal{R}$ is a map from $V^A \otimes V^B$ to $V^B \otimes V^A$ whose action is defined by (in the local theory this involves adopting some suitable gauge fixing):

$$\mathcal{R} \cdot |g> \otimes |h> = |ghg^{-1}> \otimes |g> \tag{6}$$

The braid operator encodes the braiding properties of the defects. Note that it braids the defect to the right halfway around the other defect, it implements a *half-braiding* or an interchange. To achieve a full braiding, we have to apply the *monodromy* operator which equals $\mathcal{R}^2$.
The equation for the braiding of defects $|g>$ and $|h>$ we have just discussed applies equally well to the cases of global and local symmetry breaking.

Finally, it is clear that electric modes braid trivially with each other[5]:

$$\mathcal{R}|v_1> \otimes |v_2> = |v_2> \otimes |v_1> . \tag{7}$$

Before turning to more formal aspects of braiding let us comment on the case of global symmetry. For example in a crystal the defects are defined using a Burgers vector (giving a displacement after circumventing a *dislocation*) or a Frank vector (specifying a rotation being the deficit angle after one carries a vector around a *disclination*). The first has to do with translations and the other with discrete rotations, both are elements of the discrete symmetry group of the lattice. The story is thus very similar to the local case. The essence is that the local lattice frame is changed while circumventing the defect (one speaks of "frame dragging") and that will obviously affect the propagation of local degrees of freedom, whether these are ordinary modes or other defects.

There is also a "continuum approach" to lattice defects where these are considered as singularities in the curvature and torsion of some metric space. In other words, the defects affect the space around them, which will leave its traces if one is to bring a particle around the defect. This idea has been used in crystals, and the resulting geometry is of the Riemann-Cartan type[6] where the disclinations and dislocations can be considered as singular nodal lines of curvature and torsion respectively. For an introduction, see [20] and [21]. Clearly this geometrical approach to defects assumes some suitable continuum limit to be taken, which turns the theory into an $ISO(3)$ gauge theory (just like gravity is an $SO(3,1)$ gauge theory).

Describing phases with global symmetries in terms of gauge fields leads to the analog of the Aharonov-Bohm effect in global phases. This has been studied in many phases, such as superfluid helium [22], crystals [23], and uniaxial nematic liquid crystals [24] (neglecting diffusion).

So far it is clear that for the global theory, the outcome of braiding is basically the same as in the local case. The frame dragging however is locally measurable and cannot be changed by local gauge transformations. This means that the global theory may admit additional (nonivariant) observables.

We have seen that in basically all situations the action of the braid operator on a two-particle state consisting of a defect and an ordinary (electric) excitation is

$$\mathcal{R} \cdot (|g> \otimes |v>) = (\alpha(g)|v>) \otimes |g> \tag{8}$$
$$\mathcal{R} \cdot (|v> \otimes |g>) = |g> \otimes |v> \tag{9}$$

The beauty of many Hopf algebras $\mathcal{A}$ - and the quantum double is one of

them - is that they are *quasitriangular*, which means that they are naturally endowed with a *universal R-matrix* denoted by R. R is an element of $\mathcal{A} \otimes \mathcal{A}$. It encodes the braiding of states of the irreps of $\mathcal{A}$: to braid two states, $|\phi_1>$ in Π_1 and $|\phi_2>$ in Π_2, act with R on $|\phi_1> \otimes |\phi_2>$, and then apply the flip operator τ. This gives the action of the braid operator $\mathcal{R}$:

$$\mathcal{R}(|\phi_1> \otimes |\phi_2>) = \tau \circ (\Pi_1 \otimes \Pi_2) \circ R \circ |\phi_1> \otimes |\phi_2> . \tag{10}$$

The operator τ trivially interchanges any two vectors $|\phi_1>$ and $|\phi_2>$ around:

$$\tau(|\phi_1> \otimes |\phi_2>) = |\phi_2> \otimes |\phi_1> . \tag{11}$$

The braid (or rather monodromy) operator shows up in invariant physical quantities. For example in the gauge theory case, the expression for the cross section for the nonabelian analogue of Ahoronov-Bohm scattering involving the monodromy operator was given by E. Verlinde [25]:

$$\frac{d\sigma}{d\theta} = \frac{1}{4\pi p sin^2\theta/2}[1 - Re < \psi_{in}|\mathcal{R}^2|\psi_{in}>] \tag{12}$$

where $|\psi_{in}>$ is the initial internal wavefunction of the whole system, and p the incoming momentum. For example, for the case of an electric mode $|v>$ to the left of a vortex $|g>$, we have $|\psi_{in}>= |v> \otimes |g>$. If the braiding is trivial, i.e. $\mathcal{R}^2|\psi_{in}>= |\psi_{in}>$, then there is no scattering.

The question whether this expression for the cross section is also valid in the case of global symmetries, has not been answered conclusively, see for example [26]. As far as the analysis of conceivable ways to break Hopf symmetries, the general theory applies to both situations, but in the actual application there is a precise mathematical difference between phases with local or global symmetries.

2.3 Confinement

We have already indicated that excitations which braid nontrivially with the vacuum state cannot be strictly *local* excitations, because the condensate cannot be single valued around them. Such an excitation can only be excited at the price of creating a physical halfline singularity attached to it. One has to create a configuration of a domain wall ending at a defect and therefore we say that such a defect is confined. The fate - such as confinement - of defects in phase transitions where the topology changes was discussed in relation with a particular exact sequence of homotopy mappings in [27]. We will see that the

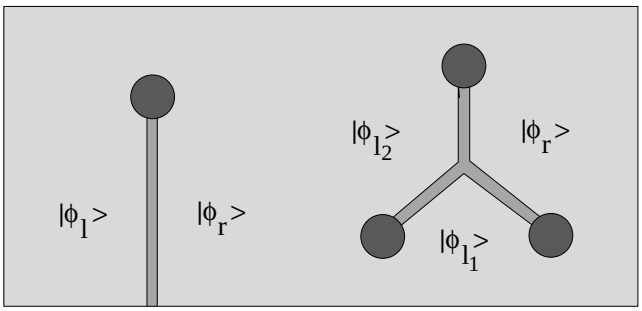

Fig. 3. A confined particle (left) and a hadronic composite (right). Confinement of particles in the ordered phase, and hadronic composites. The dark lines are singularities in the condensate and cost a finite amount of energy per unit length.

condensate and hence the vacuum vector takes a different value to the left and to the right of the domain wall. And since the domain wall emanating from the excitation costs a finite amount of energy per unit length, we have to conclude that such an excitation will be linearly *confined*. A finite energy configuration requires another confined excitation on the other end of the wall. We will call a configuration consisting of a number of confined constituents connected by a set of walls, a *hadronic composite*, in analogy with the hadrons in QCD, being the unconfined composites of confined quarks (see Figure 3). The excitations that braid trivially with the condensate are clearly *unconfined*, and they can propagate as isolated particles.

To illustrate some of these aspects in more detail, we take a look at the ubiquitous XY-model in two dimensions, and reinterpret the phase transition to the ordered state as an example where *confinement* plays a role.

Let $\theta(\vec{r})$ be the angular variable of the XY model, and $\phi = |\phi|e^{i\theta}$ the order parameter. The equation that $\theta(\vec{r})$ has to satisfy in order to minimize the free energy of the XY model is Laplace's equation in two dimensions:

$$\nabla^2\theta = 0. \tag{13}$$

There are point singular solutions $\theta_{n,\vec{r_0}}$ of (13) that correspond to a defect of charge n centered at $\vec{r_0}$. There are also other singular solutions of (13), in which the singularity is a half-line. They are labelled by a $\lambda \in \mathbb{R}$ and a vector $\vec{r_0} = (x_0, y_0)$ and the corresponding solution is given by

$$\theta_{\lambda,\vec{r_0}}(x,y) = \eta\ arctan(\frac{x-x_0}{y-y_0}) = \lambda\varphi, \tag{14}$$

with φ the polar angle.

Note that for $\lambda = n \in \mathbb{Z}$, there is indeed no line singularity. For $\lambda \notin \mathbb{Z}$, there is a line starting at $\vec{r_0}$ and going out to infinity, along which $\phi = ve^{i\theta}$

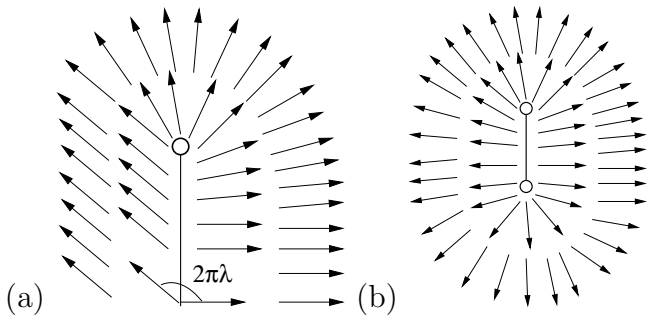

Fig. 4. Confinement of vortices of non-integer charge λ in the ordered phase of the XY-model. The arrows point in the direction of the local value of θ. In (a) we see a confined charge λ vortex in the XY model. A half-line singularity starts at the core of the defect (schematically indicated by the small circle) and goes out to infinity. It carries a finite amount of energy per unit length. (b) shows an unconfined composite of two confined charge $\frac{1}{2}$ vortices. The charge $\frac{1}{2}$ vortices are attached by a singular line. The overall configuration has charge 1.

is discontinuous (we assume $|\phi| = v$ is constant in the ordered phase). To see that there is a discontinuity, we follow a loop around $\vec{r_0}$, and notice that as we go full circle θ turns by an angle $2\pi\lambda$. If $\lambda \notin \mathbb{Z}$, θ does not return to its original value as we finish travelling along our loop (remember that θ is defined modulo 2π). Thus there is a half line singularity in θ, which implies a half line singularity in ϕ. One easily checks that this line singularity carries a finite amount of energy per unit length. Thus, if $\lambda \notin \mathbb{Z}$, the free energy of the configuration $\theta_{\lambda,\vec{r}_0}$ increases linearly with the system size. Such a wall with an end is not a topological defect in the strict sense, since it does not carry a $\Pi_1(G/H)$ charge, but its appearance can be understood from analyzing the appropriate exact homotopy sequence [27] . We will call it a vortex with non-integer charge λ, and conclude that this vortex is *confined*. It is attached to a half-line singularity which corresponds to a *domain wall*, because the "line" bounded by the defect is of one dimension lower than the dimension of the space. The usual definition implies that an excitation is confined if its energy increases linearly with the system size.

After symmetry breaking, the vortices of non-integer charge are confined. It is conceivable that there are noninteger charges in this phase, connected together by half-line singularities, such that the overall charge is integer. See Figure 4 for an example of a hadronic composite of two charge $\frac{1}{2}$ vortices.

As mentioned, the crucial characteristic of confined excitations is that the condensate takes on a different value to the left and right of the excitation. We started by condensing the order parameter field, such that it took on the value $|\phi_0>$ everywhere. Now we have an excitation such that the order parameter field takes the value $|\phi_l>$ to the left, and $|\phi_r>$ to the right of the half-line singularity connected to the excitation. To unambiguously define the vacuum state $|\phi_0>$, we must make a choice how we treat the excitations. This

we do by specifying that excitations, both confined and unconfined, enter the system from the left. Thus, we set $|\phi_0>=|\phi_r>$, and as a confined excitation comes in from the left the condensate to the left of the excitation takes on the value $|\phi_l> \neq |\phi_0>$. If an unconfined excitation comes in from the left, then $|\phi_l>=|\phi_r>=|\phi_0>$.

Imagine a particle in a state $|v>$ coming in from the left. We want to know whether $|v>$ is confined of not, i.e. we need to know if $|\phi_l>=|\phi_r>$, in which case $|v>$ is unconfined. This is where braiding comes in: $|\phi_l>$ is the outcome of half-braiding $|\phi_r>$ counterclockwise around $|v>$. Thus, to find out whether $|v>$ is confined, we braid the condensate around the excitation, using the braid operator $\mathcal{R}$. For $|v>$ to be unconfined, the condensate $|\phi_r>$ has to braid trivially around $|v>$, both clockwise and counterclockwise. We must check that both braidings are trivial, because the half-line singularity could run along the positive or the negative x-axis in fig. 4. This leads to the following test of whether an excitation is confined or unconfined:

$$
\begin{array}{l}
|v> \text{ is unconfined} \\
\qquad \Longleftrightarrow \\
\begin{cases}
\mathcal{R}\cdot(|v>\otimes|\phi_r>) = |\phi_r>\otimes|v> \\
\mathcal{R}^{-1}\cdot(|v>\otimes|\phi_r>) = |\phi_r>\otimes|v> \,.
\end{cases}
\end{array}
$$

Let us apply this criterion to the vortices in the XY model, to see which ones are unconfined. Denote a vortex of charge λ by $|\lambda>$. We set $|\phi_r>=v$. Then $|\phi_r>$ gets frame dragged as we braid it counterclockwise around $|\lambda>$, and picks up a phase factor $e^{i2\pi\lambda}$. Clockwise braiding is trivial, since we've chosen the convention that the clockwise braiding of an electric mode to the right of a defect is trivial[7]. Thus

$$
\begin{aligned}
&\mathcal{R}\cdot(|\lambda>\otimes|\phi_r>) = e^{i2\pi\lambda}|\phi_r>\otimes|\lambda> \\
&\mathcal{R}^{-1}\cdot(|\lambda>\otimes|\phi_r>) = |\phi_r>\otimes|\lambda>,
\end{aligned}
$$

and a vortex is unconfined $\iff$ $\lambda\in\mathbb{Z}$. Thus the defects are precisely the unconfined vortices.

In the high temperature phase, the "electric" excitations are irreps of the electric group $U(1)$. If we also consider projective irreps, then the electric excitations are irreps of the universal covering group $\widetilde{U(1)}=\mathbb{R}$ of $U(1)$. The irreps are denoted by ρ_r, they are labelled by an $r\in\mathbb{R}$. All these irreps braid trivially with the condensate $|\phi_r>=v$, because the condensate is in a state $|\phi_r>$ in an electric irrep, and electric irreps braid trivially with each other. Thus in the ordered phase all the electric excitations are unconfined. The

situation is different from the situation where defects condense as is discussed in [4].

So indeed the phase transition from the high temperature to the ordered phase in the XY-model can in this sense be described as a confining transition.

3 Hopf symmetry breaking: the formalism

In this section we will first study symmetry breaking for a phase described by a general Hopf algebra $\mathcal{A}$. Then we will specialize to the cases where $\mathcal{A}$ is a modified quantum double: $\mathcal{A} = F(H_m) \times \mathbb{C}H_{el}$. The quantum double $D(H)$ is a special case of the modified quantum double, with $H_m = H_{el} = H$.

3.1 The criterion for symmetry breaking

We consider a physical system whose excitations (or particles) are labeled by irreps of a quasitriangular Hopf algebra $\mathcal{A}$. We say that a condensate forms in a state $|\phi_r>$ of an irrep Π_ϕ of $\mathcal{A}$, so that our ground state is a state filled with the particles in the state $|\phi_r>$. We now have to define the *residual symmetry algebra* $\mathcal{T}_r$, which is a symmetry algebra with the property that the excitations of the new ground state form irreps of $\mathcal{T}_r$.

The residual symmetry algebra $\mathcal{T}_r$ consists of the operators that are 'well defined with respect to the condensate'. Before we explain what that means, let us first consider the case of a system whose symmetry is an ordinary group G. G is broken spontaneously to H by a condensate in a state $|\phi_r>$ of an irrep of G. H is the stabilizer of $|\phi_r>$, i.e. the subset of symmetry transformations in G that leave the condensate invariant. And because they leave the condensate invariant we can implement these transformations on excitations of the condensate. Thus they act on the excitations, and since by definition they also commute with the Hamiltonian, they transform low energy excitations into excitations of same energy, which can therefore be organized in irreps of H .

Now consider a particle in state $|v>$ of an irrep of the original symmetry G. If we fuse $|v>$ with $|\phi_r>$, and act on the outcome $|v>|\phi_r>$ of this fusion with an $h \in H$, the result is the same as when we act on $|v>$ with h first, and then fuse the outcome with $|\phi_r>$. This follows from

$$h \cdot (|v> \otimes |\phi_r>) = (h \cdot |v>) \otimes (h \cdot |\phi_r>) = (h \cdot |v>) \otimes |\phi_r>, \qquad (15)$$

since $h \cdot |\phi_r> = |\phi_r>$. Thus we can also define the residual symmetry group

H to be the set of transformations that *are insensitive to fusion with the condensate*. Whether a particle $|v>$ in an irrep of G fuses with the condensate or not, the action of H is the same. Thus we define *residual symmetry operators* to be operators that are not affected by fusion with the condensate. When the symmetry is spontaneously broken from G to H, the residual symmetry operators are the elements of H.

This definition of residual symmetry operator can be carried over to the case of a condensate $|\phi_r>$ in an irrep of a Hopf symmetry $\mathcal{A}$. The residual symmetry operators are the operators that are not affected by fusion with the condensate. The residual symmetry operators form a subalgebra of $\mathcal{A}$, which we call the *residual symmetry algebra* $\mathcal{T}_r$. There is a subtlety in the definition of $\mathcal{T}_r$: an operator $a \in \mathcal{A}$ is an element of $\mathcal{T}_r$ if its action on any particle $|v>$ in any irrep of $\mathcal{A}$ is the same whether $|v>$ has fused with the condensate $|\phi_r>$ or not. But we must specify whether $|v>$ fuses with $|\phi_r>$ from the left or the right! Namely, in the systems we are considering $|v> \otimes |\phi_r>$ and $|\phi_r> \otimes |v>$ are not (necessarily) the same state, because of the possibility of nontrivial braiding. Thus we have to fix a convention. This convention is set by our earlier choice of having all particles come in from the left. Remember that we had to make this choice because some excitations of the condensate may be confined. Thus we define $\mathcal{T}_r$ as follows: $a \in \mathcal{T}_r \iff$ for any particle $|v>$ in any irrep of $\mathcal{A}$, we have

$$a \cdot (|v> \otimes |\phi_r>) = (a \cdot |v>) \otimes |\phi_r> . \tag{16}$$

Since $|v>$ and $|\phi_r>$ are states of particles of $\mathcal{A}$, their fusion is set by the coproduct Δ of $\mathcal{A}$:

$$a \cdot (|v> \otimes |\phi_r>) = (\Pi \otimes \Pi_\phi) \circ \Delta(a)(|v> \otimes |\phi_r>), \tag{17}$$

where $|v>$ is in the irrep Π of $\mathcal{A}$, and $|\phi_r>$ in the irrep Π_ϕ of $\mathcal{A}$. Since this equation has to hold for all vectors $|v>$ in all irreps Π of $\mathcal{A}$, it is equivalent to[8]

$$(1 \otimes \Pi_\phi)\Delta(a)(1 \otimes |\phi_r>) = a \otimes |\phi_r> . \tag{18}$$

$\mathcal{T}_r$ consists of all operators that satisfy this criterion. We will now prove that $\mathcal{T}_r$ is a subalgebra of $\mathcal{A}$. We also prove two more properties of $\mathcal{T}_r$ which will play a role later on.

Lemma 1 *The elements of a finite dimensional Hopf algebra $\mathcal{A}$ that satisfy*

$$(1 \otimes \Pi_\phi)\Delta(a)(1 \otimes |\phi_r>) = a \otimes |\phi_r> \tag{19}$$

form a subalgebra $\mathcal{T}_r$ of $\mathcal{A}$ that satisfies:

(1) $\Delta(\mathcal{T}_r) \subset \mathcal{A} \otimes \mathcal{T}_r$
(2) The elements of $\mathcal{T}_r$ leave tensor products of the vacuum invariant.

PROOF.

That $\mathcal{T}_r$ is an algebra follows from the fact that Δ is an algebra morphism.

(1) Using the definition of $\mathcal{T}_r$ and the coassociativity of the coproduct,

$$\begin{aligned}
&(1 \otimes 1 \otimes \Pi_\phi)(1 \otimes \Delta)\Delta(a)(1 \otimes |\phi_r>) = \\
&= (1 \otimes 1 \otimes \Pi_\phi)(\Delta \otimes 1)\Delta(a)(1 \otimes |\phi_r>) = \\
&= (\Delta \otimes 1)(1 \otimes \Pi_\phi)\Delta(a)(1 \otimes |\phi_r>) \\
&= (\Delta \otimes 1)(a \otimes |\phi_r> = \sum_{(a)} a^{(1)} \otimes a^{(2)} \otimes |\phi_r> \\
&\Rightarrow \sum_{(a)} a^{(1)} \otimes (\Delta(a^{(2)}) \cdot (1 \otimes |\phi_r>)) = \sum_{(a)} a^{(1)} \otimes a^{(2)} \otimes |\phi_r> \\
&\Rightarrow \Delta(a^{(2)}) \cdot (1 \otimes |\phi_r>) = a^{(2)} \otimes \phi \\
&\Rightarrow a^{(2)} \in \mathcal{T}_r \\
&\Rightarrow \Delta(a) \in \mathcal{A} \otimes \mathcal{T}_r
\end{aligned}$$

(2) We just proved that $(1 \otimes \Pi_\phi)\Delta(a^{(2)})(1 \otimes |\phi_r>) = a^{(2)} \otimes |\phi_r>$. Using this, we get

$$\begin{aligned}
&(1 \otimes \Pi_\phi \otimes \Pi_\phi)(1 \otimes \Delta)\Delta(a)(1 \otimes |\phi_r> \otimes |\phi_r>) \\
&= \sum_{(a)} a^{(1)} \otimes ((\Pi_\phi \otimes 1)(1 \otimes \Pi_\phi)\Delta(a^{(2)})(1 \otimes |\phi_r>)(|\phi_r> \otimes 1)) \\
&= \sum_{(a)} a^{(1)} \otimes ((\Pi_\phi \otimes 1)(a^{(2)} \otimes |\phi_r>)(|\phi_r> \otimes 1)) \\
&= ((1 \otimes \Pi_\phi)\Delta(a)(1 \otimes |\phi_r>)) \otimes |\phi_r> \\
&= a \otimes |\phi_r> \otimes |\phi_r>
\end{aligned}$$

We call $\mathcal{T}_r$ the *right residual symmetry algebra.*
What if we condense the sum of two vectors in different irreps, $|\phi_1> + |\phi_2>$? According to (18), $a \in \mathcal{A}$ is part of the right residual symmetry algebra $\mathcal{T}_r$ of $|\phi_1> + |\phi_2>$ if

$$\begin{aligned}
(1 \otimes \Pi_{\phi_1})(\Delta(a))(1 \otimes |\phi_1>) + (1 \otimes \Pi_{\phi_2})(\Delta(a))(1 \otimes |\phi_2>) \\
= a \otimes |\phi_1> + a \otimes |\phi_2> .
\end{aligned} \tag{20}$$

Since we are dealing with irreps, the only way to get an equality is by equating the first terms on the left-hand side and right-hand side of this equation, and

the last terms, separately. So $\mathcal{T}_r = (\mathcal{T}_r)_1 \cap (\mathcal{T}_r)_2$, the intersection of the right residual symmetry algebras of $|\phi_1>$ and $|\phi_2>$. Therefore, we need only treat the condensation of vectors in one irrep, since we can then take intersections of the right residual symmetry algebras of condensates in different irreps to get the right residual symmetry algebra for any condensate.

We will now specialize to the case where $\mathcal{A}$ is a modified quantum double:

$$\mathcal{A} = F(H_m) \times \mathbb{C}H_{el},$$

where H_m and H_{el} are groups (see Appendix A).

The irreps of $\mathcal{A} = F(H_m) \times \mathbb{C}H_{el}$ are labelled by an orbit A in H_m under the action of H_{el}, and an irrep of the normalizer N_A of a preferred element $g_A \in A$.

In Appendix A we show how to write elements of $F(H_m) \times \mathbb{C}H_{el}$ as functions $f \in F(H_m \times H_{el})$. In this notation the derivations to come are more elegant. Consider symmetry breaking by a condensate $|\phi_r>$ in an irrep Π^A_α of $\mathcal{A}$.

Lemma 2 *Take $f \in F(H_m \times H_{el})$. Then*

$$\begin{array}{rl} f \in \mathcal{T}_r \iff & f(x_1(x \cdot g_A), y_1)|\phi_r(y_1^{-1}x)> = f(x_1, y_1)|\phi_r(x)> \\ \forall & x_1 \in H_m, x, y_1 \in H_{el} \end{array} \tag{21}$$

PROOF. We use the formulae in Appendix A.

$$\begin{aligned} & (id \otimes \Pi^A_\alpha)\Delta(f)(1 \otimes |\phi_r(x)>) = f \otimes |\phi_r(x)> \\ \iff & 1 \otimes \sum_{z \in H_{el}} \Delta(f)(x_1, y_1; x \cdot g_A, z)|\phi_r(z^{-1}x)> = f(x_1, y_1) \otimes |\phi_r(x)> \\ \iff & 1 \otimes \sum_{z \in H_{el}} f(x_1(x \cdot g_A), y_1)\delta_{y_1}(z)|\phi_r(z^{-1}x)> = f(x_1, y_1) \otimes |\phi_r(x)> \\ \iff & 1 \otimes f(x_1(x \cdot g_A), y_1)|\phi_r(y_1^{-1}x)> = f(x_1, y_1) \otimes |\phi_r(x)> \end{aligned}$$

$\mathcal{T}_r$ was obtained by condensing $|\phi_r>$ to the right of our system. If we choose to condense $|\phi_r>$ to the left, we get another residual symmetry algebra:

$$\mathcal{T}_l = \{a \in \mathcal{A} : (\Pi_\phi \otimes id)\Delta(a)(|\phi_r> \otimes 1) = |\phi_r> \otimes a\}. \tag{22}$$

We call $\mathcal{T}_l$ the *left residual symmetry algebra.*

It is interesting to compare this criterion to the one in [2]. There, the *residual symmetry algebra* is denoted by $\mathcal{T}$, and is defined as the largest Hopf subalgebra of $\mathcal{A}$ whose elements satisfy

$$a \cdot |\phi_r >= \varepsilon(a)|\phi_r > . \tag{23}$$

The motivation for this criterion is: The residual symmetry operators act on the condensate like the vacuum irrep ε does.

The important difference between $\mathcal{T}$ and $\mathcal{T}_r$ is that we don't require $\mathcal{T}_r$ to be a Hopf algebra. In particular, we don't expect the residual symmetry to have a coproduct. In fact there is a physical reason why we don't need to have a coproduct for $\mathcal{T}_r$. We have chosen the condensate to be $|\phi_r >$ on the right. If we bring in a confined excitation from the left, the condensate will be in a state $|\phi_l >$ to the left of this excitation. Now consider a second particle coming in from the left, it sees the condensate $|\phi_l >$, and it is therefore an excitation associated with the residual symmetry of $|\phi_l >$, which needn't be equal to the residual symmetry of $|\phi_r >$! Thus, we have to keep track of the *ordering* of the particles, i.e. it is crucial to know in which order we brought in the particles from the left. We will see that - not surprisingly - this translates into the absence of a coproduct in $\mathcal{T}_r$, namely we can't simply fuse irreps of $\mathcal{T}_r$. Before we can discuss this in more detail, we need to take a closer look at the structure of the residual symmetry algebras.

3.2 Relationship between $\mathcal{T}$, $\mathcal{T}_r$ and $\mathcal{T}_l$

We will now establish a number of interesting connections between $\mathcal{T}$, $\mathcal{T}_r$ and $\mathcal{T}_l$. All the operators in $\mathcal{T}_r$ and $\mathcal{T}_l$ satisfy (23):

$$a \cdot |\phi_r >= \varepsilon(a)|\phi_r > . \tag{24}$$

Thus the operators of $\mathcal{T}_r$ and $\mathcal{T}_l$ act on the condensate like the vacuum irrep ε does, just as the operators of $\mathcal{T}$ do. At the same time, we have the inclusions

$$\mathcal{T} \subset \mathcal{T}_r \quad \mathcal{T} \subset \mathcal{T}_l. \tag{25}$$

The left and right residual symmetry algebras contains $\mathcal{T}$. We prove these statements in the following lemma.

Lemma 3 *$\mathcal{T}_r, \mathcal{T}_l$ and $\mathcal{T}$ satisfy the following:*

(1) All elements of $\mathcal{T}_r$ and of $\mathcal{T}_l$ satisfy (23):

$$a \cdot |\phi_r >= \varepsilon(a)|\phi_r > \tag{26}$$

(2) $\mathcal{T} \subset \mathcal{T}_r \cap \mathcal{T}_l$

PROOF.

(1) $a \in \mathcal{T}_r$ implies:

$$\begin{aligned}
&(id \otimes \Pi_\phi)\Delta(a)1 \otimes |\phi_r >= a \otimes |\phi_r > \\
\Rightarrow\ &(\varepsilon \otimes 1)(1 \otimes \Pi_\phi)\Delta(a)1 \otimes |\phi_r >= \varepsilon(a) \otimes |\phi_r > \\
\Rightarrow\ &(1 \otimes \Pi_\phi)(\varepsilon \otimes 1)\Delta(a)1 \otimes |\phi_r >= \varepsilon(a) \otimes |\phi_r > \\
\Rightarrow\ &1 \otimes \Pi_\phi(a)|\phi_r >= \varepsilon(a) \otimes |\phi_r > \\
\Rightarrow\ &\Pi_\phi(a)|\phi_r >= \varepsilon(a)|\phi_r >,
\end{aligned}$$

where we used one the axioms of a Hopf algebra: $(\varepsilon \otimes id)\Delta(a) = 1 \otimes a$. This proves the claim for $\mathcal{T}_r$. The proof for $\mathcal{T}_l$ is analogous.

(2) $\mathcal{T}$ is a Hopf algebra, so

$$\begin{aligned}
&a \in \mathcal{T} \\
&\Rightarrow\ \Delta(a) = \sum_{(a)} a^{(1)} \otimes a^{(2)} \in \mathcal{T} \otimes \mathcal{T} \\
&\Rightarrow\ a^{(2)} \in \mathcal{T} \Rightarrow a^{(2)} \cdot |\phi_r >= \varepsilon(a^{(2)})|\phi_r > \\
&\Rightarrow\ (id \otimes \Pi_\phi)\Delta(a)(1 \otimes |\phi_r >) = (id \otimes \varepsilon)\Delta(a)(1 \otimes |\phi_r >) = a \otimes |\phi_r > \\
&\Rightarrow\ a \in \mathcal{T}_r.
\end{aligned}$$

This proves $\mathcal{T} \subset \mathcal{T}_r$. The proof that $\mathcal{T} \subset \mathcal{T}_l$ is analogous.

$\mathcal{T}_r$ and $\mathcal{T}_l$ are not necessarily Hopf algebras, while $\mathcal{T}$ is a Hopf algebra by definition. It turns out that $\mathcal{T}_r$ is a Hopf algebra $\iff$ $\mathcal{T}_r = \mathcal{T}$! Similarly, $\mathcal{T}_l$ is a Hopf algebra $\iff$ $\mathcal{T}_l = \mathcal{T}$. Also, $\mathcal{T}_r = \mathcal{T}$ $\iff$ $\mathcal{T}_r = \mathcal{T}_l$. Thus $\mathcal{T}_r$ and $\mathcal{T}_l$ are interesting extensions of $\mathcal{T}$: if they are equal to each other, then they are equal to $\mathcal{T}$. Thus, the difference between $\mathcal{T}_r$ and $\mathcal{T}_l$ is a measure of the departure of $\mathcal{T}_r$ (and $\mathcal{T}_l$) from being a Hopf algebra.

We need one assumption about $\mathcal{A}$ to prove these propositions: the antipode S of $\mathcal{A}$ must satisfy $S^2 = id$. Modified quantum doubles, for example, satisfy this property.

First we prove a little lemma.

Lemma 4 *If the antipode S of $\mathcal{A}$ satisfies $S^2 = id$, then $S(\mathcal{T}_l) = \mathcal{T}_r$ and $S(\mathcal{T}_r) = \mathcal{T}_l$.*

PROOF. According to $(A.6)$, and using $S^2 = id$:

$$\Delta^{op} \circ S = (S \otimes S) \circ \Delta$$
$$\Rightarrow \Delta^{op} = (S \otimes S) \circ \Delta \circ S$$

Say $a \in \mathcal{T}_l$, then using the last equation

$$(\Pi_\phi \otimes 1)\Delta(a)|\phi_r > \otimes 1 = |\phi_r > \otimes a$$
$$\Rightarrow (1 \otimes \Pi_\phi)\Delta^{op}(a)1 \otimes |\phi_r >= a \otimes |\phi_r >$$
$$\Rightarrow (1 \otimes \Pi_\phi)(S \otimes S)\Delta(S(a))1 \otimes |\phi_r >= (a \otimes 1)(1 \otimes |\phi_r >)$$

Apply $S \otimes S$ to left the and right, and use $S^2 = id$ and $S(1) = 1$ to get

$$\Rightarrow (1 \otimes \Pi_\phi)\Delta(S(a))1 \otimes |\phi_r >= S(a) \otimes |\phi_r >$$
$$\Rightarrow (1 \otimes \Pi_\phi)\Delta(S(a))1 \otimes |\phi_r >= S(a) \otimes |\phi_r >$$
$$\Rightarrow S(a) \in \mathcal{T}_r$$

So $S(\mathcal{T}_l) \subseteq \mathcal{T}_r$. Similarly we can prove that $S(\mathcal{T}_r) \subseteq \mathcal{T}_l$. Since S is invertible, we have $dim(S(\mathcal{T}_r)) = dim(\mathcal{T}_r)$ and $dim(S(\mathcal{T}_l)) = dim(\mathcal{T}_l)$, where dim is the dimension as a vector space. Therefore, using $S(\mathcal{T}_l) \subseteq \mathcal{T}_r$ and $S(\mathcal{T}_r) \subseteq \mathcal{T}_l$, we get

$$dim(S(\mathcal{T}_l)) \leq dim(\mathcal{T}_r) = dim(S(\mathcal{T}_r)) \leq dim(\mathcal{T}_l) = dim(S(\mathcal{T}_l)). \tag{27}$$

Thus $dim(\mathcal{T}_r) = dim(S(\mathcal{T}_l))$. Since $S(\mathcal{T}_l) \subseteq \mathcal{T}_r$, we must have $S(\mathcal{T}_l) = \mathcal{T}_r$. Applying S to both side of this equation, we get $\mathcal{T}_l = S(\mathcal{T}_r)$.

This lemma states that the antipode S brings us from $\mathcal{T}_r$ to $\mathcal{T}_l$, and back. In appendix A we show how S is used to construct the antiparticle or conjugate irrep of a given irrep. Thus, going from $\mathcal{T}_r$ to $\mathcal{T}_l$ is tantamount to replacing all particles by their antiparticles!

Using lemma 4, we can prove all our propositions about the relationships between $\mathcal{T}$, $\mathcal{T}_r$ and $\mathcal{T}_l$.

Proposition 5 *For $\mathcal{A}$ an n-dimensional Hopf algebra whose antipode S satisfies $S^2 = id$, we have*

$$(1)\mathcal{T}_r = \mathcal{T}_l \iff (2)\ \mathcal{T}_r \text{ is a Hopf algebra} \iff (3)\mathcal{T}_r = \mathcal{T} \iff (4)\mathcal{T}_l \subseteq \mathcal{T}_r$$

PROOF.

- (1) $\Rightarrow$ (2)
  We assume $\mathcal{T}_r = \mathcal{T}_l$. Take $a \in \mathcal{T}_r = \mathcal{T}_l$. To prove that $\mathcal{T}_r$ is a Hopf subalgebra of $\mathcal{A}$, we need to prove three things:

$$1 \in \mathcal{T}_r, \quad \Delta(a) \in \mathcal{T}_r \otimes \mathcal{T}_r, \quad S(\mathcal{T}_r) \subset \mathcal{T}_r. \tag{28}$$

  The first demand is trivial, because $\Delta(1) = 1 \otimes 1$, so that

$$(1 \otimes \Pi_\phi)\Delta(1)(1 \otimes |\phi_r >) = 1 \otimes \Pi_\phi(1)|\phi_r >= 1 \otimes |\phi_r > \Rightarrow 1 \in \mathcal{T}_r. \tag{29}$$

  For the second demand: since $a \in \mathcal{T}_r = \mathcal{T}_l$, we have $\Delta(a) \in \mathcal{T}_l \otimes \mathcal{A} = \mathcal{T}_r \otimes \mathcal{A}$, and $\Delta(a) \in \mathcal{A} \otimes \mathcal{T}_r$. Choose a basis $\{r_i\}_{1 \le i \le k}$ of $\mathcal{T}_r$, and a basis $\{a_j\}_{k+1 \le j \le n}$ of $\mathcal{T}_r^\perp$. Then

$$\begin{aligned}
&\Delta(a) \in \mathcal{T}_r \otimes \mathcal{A} \Rightarrow \Delta(a) = \sum r_i \otimes a_i' \\
&\text{Write } a_i' \text{ out in terms of the bases } \{r_i\} \text{ of } \mathcal{T}_r \text{ and } \{a_j\} \text{ of } \mathcal{T}_r^\perp: \\
&a_i' = B_{ij} r_j + C_{ik} a_k \quad B_{ij}, C_{ik} \in \mathbb{C} \\
&\Rightarrow \Delta(a) = \sum r_i \otimes B_{ij} r_j + \sum r_i \otimes C_{ik} a_k \\
&\Delta(a) \in \mathcal{A} \otimes \mathcal{T}_r \Rightarrow C_{ik} a_k = 0 \\
&\Rightarrow \Delta(a) = \sum r_i \otimes B_{ij} r_j \in \mathcal{T}_r \otimes \mathcal{T}_r
\end{aligned}$$

  Now for $S(\mathcal{T}_r) \subseteq \mathcal{T}_r$. Using lemma 4, $S(\mathcal{T}_r) = \mathcal{T}_l = \mathcal{T}_r$.
  Thus $\mathcal{T}_r$ is a Hopf subalgebra of $\mathcal{A}$.
- (2) $\Rightarrow$ (3)
  According to lemma 3, we have $\mathcal{T} \in \mathcal{T}_r$, and all $a \in \mathcal{T}_r$ satisfy $\Pi_\phi(a)|\phi_r >= \epsilon(a)|\phi_r >$. $\mathcal{T}$ was defined as the largest Hopf subalgebra of $\mathcal{A}$ whose elements satisfy $\Pi_\phi(a)|\phi_r >= \epsilon(a)|\phi_r >$, so if $\mathcal{T}_r$ is a Hopf algebra we must have $\mathcal{T}_r = \mathcal{T}$.
- (3) $\Rightarrow$ (1)
  We already proved that $\mathcal{T} \subseteq \mathcal{T}_l$. Thus if we assume $\mathcal{T} = \mathcal{T}_r$ we have $\mathcal{T}_r \subseteq \mathcal{T}_l$.
  Since $\mathcal{T}_r = \mathcal{T}$, $\mathcal{T}_r$ is a Hopf algebra. Thus $S(\mathcal{T}_r) \subseteq \mathcal{T}_r$. From lemma 4, we know that $S(\mathcal{T}_r) = \mathcal{T}_l$. Thus $\mathcal{T}_l \subseteq \mathcal{T}_r$.
  Done: $\mathcal{T}_r = \mathcal{T}_l$.
- (1) $\Rightarrow$ (4)
  Obvious.
- (4) $\Rightarrow$ (1)
  Lemma 4 taught us that $\mathcal{T}_r = S(\mathcal{T}_l)$ and $S(\mathcal{T}_r) = \mathcal{T}_l$. Apply S to the left and right of $\mathcal{T}_l \subseteq \mathcal{T}_r$ to obtain $\mathcal{T}_r \subseteq \mathcal{T}_l$. Done: $\mathcal{T}_r = \mathcal{T}_l$.

3.3 $\mathcal{T}_r$ and $\mathcal{T}_l$: Hopf or not?

We are interested in finding out which condensates yield a right residual symmetry algebra $\mathcal{T}_r$ that is a Hopf algebra. From proposition 5 we know that $\mathcal{T}_r$ is a Hopf algebra $\iff$ $\mathcal{T}_l$ is the same Hopf algebra.

As a rather general case, consider a phase with a modified quantum double as its Hopf symmetry: $\mathcal{A} = F(H_m) \times \mathbb{C}H_{el}$. We can write elements of $\mathcal{A}$ as functions $f \in F(H_m \times H_{el})$ (see appendix A). Now condense $|\phi_r>$ in an irrep Π^A_α of $\mathcal{A}$. We saw in lemma 2 that a function $f \in F(H_m \times H_{el})$ is an element of $\mathcal{T}_r$ if it satisfies

$$f(x_1(x \cdot g_A), y_1)|\phi_r(y_1^{-1}x) >= f(x_1, y_1)|\phi_r(x) > \quad \begin{array}{l} \forall\, x_1 \in G, \\ x, y_1 \in H_{el} \end{array}$$

where g_A is the preferred element of A.

Analogously to the derivation of lemma 2, we can prove that the functions f in $\mathcal{T}_l$ are precisely those f that satisfy

$$f((x \cdot g_A)x_1, y_1)|\phi_r(y_1^{-1}x) >= f(x_1, y_1)|\phi_r(x) > \quad \begin{array}{l} \forall\, x_1 \in G, \\ \forall\, x, y_1 \in H_{el}. \end{array} \tag{30}$$

Proposition 5(4) tells us that proving that $\mathcal{T}_r$ is a Hopf algebra is equivalent to proving that $\mathcal{T}_l \subseteq \mathcal{T}_r$. Thus, to prove that $\mathcal{T}_r$ is a Hopf algebra, we must prove that if a function f satisfies (30), it automatically satisfies (21):

$$\begin{array}{l} \forall x_1 \in H_m, \forall x, y_1 \in H_{el} : \\ \quad f(x_1 x \cdot g_A, y_1)|\phi_r(y_1^{-1}x) >= f(x_1, y_1)|\phi_r(x) > \\ \Rightarrow f((x \cdot g_A)x_1, y_1)|\phi_r(y_1^{-1}x) >= f(x_1, y_1)|\phi_r(x) > \end{array} \tag{31}$$

This implication is automatically satisfied if H_m is an abelian group, because then $x \cdot g_A$ and x_1 commute. Thus, if the magnetic group H_m is abelian, $\mathcal{T}_r$ is necessarily a Hopf algebra.

Also, if g_A is in the center of H_m, and is acted on trivially by all of H_{el}, then (31) is satisfied. Electric condensates are an example, since $g_A = e$ for electric condensates. e is in the center of H_m, and H_{el} acts trivially on e.

If the original phase is a $D(H)$ phase, then we have similar results: $\mathcal{T}_r$ is a Hopf algebra if H is abelian, or if g_A is in the center of H. We needn't demand

that all of H acts trivially on g_A, since this immediately follows from g_A being in the center of H.

3.4 Requirement on the condensate $|\phi_r>$

If the condensate is $|\phi_r>$, then our ground state is filled with the particles in the state $|\phi_r>$. We know that if an excitation of this ground state braids nontrivially with the condensate, then it is connected to a domain wall which costs a finite amount of energy per unit length. Thus, if $|\phi_r>$ were to braid nontrivially with itself, it wouldn't make sense to think of a condensate. Thus we require of our condensate $|\phi_r>$ that it braid trivially with itself:

$$\mathcal{R} \circ (|\phi_r> \otimes |\phi_r>) = |\phi_r> \otimes |\phi_r> . \tag{32}$$

Note that we are braiding indistinguishable particles, and if $|\phi_r>$ has spin s, then the braiding picks up an extra phase factor $e^{i2\pi s}$. The spin factor should be taken into account when verifying the trivial self braiding condition.

Recently fermionic condensates have received considerable theoretical and experimental attention[28], and we could definitely treat those as well with our methods. We may then relax the trivial self braiding condition exactly because identical fermions don't braid trivially with each other: they pick up a minus sign under half-braiding.

3.5 Unconfined excitations and the algebra $\mathcal{U}$

3.5.1 The conditions of trivial braiding

Now that we've learned how to derive $\mathcal{T}_r$, we want to study the unconfined excitations of $\mathcal{T}_r$. We found a criterion for determining whether an excitation was confined: If the excitation doesn't braid trivially with the condensate $|\phi_r>$, then it is confined.

The condensate is in a state $|\phi_r>$ of an irrep Π_ϕ of $\mathcal{A}$. Now consider an excitation of the ground state, sitting in an irrep Ω of $\mathcal{T}_r$. Since the universal R matrix $R = \sum_{(R)} R^{(1)} \otimes R^{(2)} \in \mathcal{A} \otimes \mathcal{A}$, we cannot simply act with R on states in the tensor product representation $\Pi_\phi \otimes \Omega$, because we can only act with elements of $\mathcal{T}_r$ on states of Ω. We need a projection P of $\mathcal{A}$ onto $\mathcal{T}_r$, so that we can act with $(id \otimes P)R$ on states of $\Pi_\phi \otimes \Omega$. If $\mathcal{A}$ has an inner product, then we can use this inner product to define the projection P. Take an orthonormal basis $\{a_i\}$ of $\mathcal{T}_r$, and an orthonormal basis[9] $\{b_j\}$ of $\mathcal{T}_r^\perp$. Together, the a_i and

b_j form an orthonormal basis of $\mathcal{A}$, such that the a_i span $\mathcal{T}_r$, and for all i and j we have [10]

$$(a_i, b_j) = 0, \tag{33}$$

where (a, b) denotes the inner product between a and b. Now take any $a \in \mathcal{A}$, and write

$$a = \sum_{(a_i)} (a_i, a)a + \sum_{(b_j)} (b_j, a)b_j \equiv Pa + (1 - P)a. \tag{34}$$

Thus we have our projection: $Pa = \sum_{(a_i)} (a_i, a)a \in \mathcal{T}_r$. It is known that this projection is in fact independent of our choice of basis: given a vector space with an inner product, the perpendicular projection onto a vector subspace is uniquely defined.

Modified quantum doubles $F(H_m) \times \mathbb{C}H_{el}$ come equipped with an inner product (A.28):

$$(P_g h, P_{g'} h') = \delta_{g,g'} \delta_{h,h'}. \tag{35}$$

We use this inner product to define the projection operator $P : \mathcal{A} \to \mathcal{T}_r$, just as we discussed above. Now we can define the braiding of a state $|v>$ of Ω with $|\phi_r>$:

$$\text{Counterclockwise} : \tau \circ (\Omega \otimes \Pi^A_\alpha)(P \otimes id)R(|v> \otimes |\phi_r>) \tag{36}$$

$$\text{Clockwise} \quad : \tau \circ (\Omega \otimes \Pi^A_\alpha)(P \otimes id)R^{-1}_{21}(|v> \otimes |\phi_r>) \tag{37}$$

where in the second line we used: $\mathcal{R}^{-1} = \tau \circ R^{-1}_{21}$.

We want to find out which irreps Ω of $\mathcal{T}_r$ braid trivially with the condensate, thus the braiding has to be trivial for all $|v>$ in the vector space on which Ω acts. We still need a definition of "trivial braiding". A natural definition is: Ω braids trivially with the condensate if it braids just like the vacuum irrep ε does. This definition immediately implies that the vacuum irrep ε is unconfined, since obviously ε braids like ε does. Thus the conditions for trivial braiding of an irrep Ω of $\mathcal{T}_r$ with the condensate $|\phi_r>$ living in the irrep Π^A_α of $F(H_m) \times \mathbb{C}H_{el}$ become

$$(\Omega \otimes \Pi^A_\alpha)(P \otimes 1)R(1 \otimes |\phi_r>) = (\Omega(1)\epsilon \otimes \Pi^A_\alpha)(P \otimes 1)R(1 \otimes |\phi_r>) \tag{38}$$

$$(\Omega \otimes \Pi^A_\alpha)(P \otimes id)R^{-1}_{21}\ (1 \otimes |\phi_r>) = \\ = (\Omega(1)\varepsilon \otimes \Pi^A_\alpha)(P \otimes id)R^{-1}_{21}(1 \otimes |\phi_r>). \tag{39}$$

$\Omega(1)$ is an $n \times n$ unit matrix, where n is the dimensionality of the irrep Ω.

If these equations are satisfied, then we can replace $1 \otimes |\phi_r>$ in these equations by $|v> \otimes |\phi_r>$ for any state $|v>$ of Ω. Thus if Ω satisfies the trivial braiding conditions (38) and (39), then all the states $|v>$ of Ω braid trivially with $|\phi_r>$.

An irrep Ω that satisfies these two equations is said to braid trivially with the condensate. If an irrep doesn't braid trivially with the condensate, it is a confined excitation, attached to a physical string that goes out to infinity which costs a finite amount of energy per unit length.

3.5.2 The unconfined symmetry algebra $\mathcal{U}$

The trivial braiding equations (38) and (39) divide the irreps of $\mathcal{T}_r$ into confined and unconfined irreps. We cannot simply take the tensor product of irreps of $\mathcal{T}_r$, since $\mathcal{T}_r$ isn't a Hopf algebra. The reason for the absence of a coproduct is the presence of confined excitations. The condensate to the right of a confined excitation takes on the value $|\phi_r>$, while it takes on a different value $|\phi_l>$ to the left of the excitation. Thus particles coming in from the left see a different condensate: they are excitations of the residual symmetry algebra of $|\phi_l>$.

The situation is actually a little more complicated, because the value $|\phi_l>$ of the condensate to the left of a state $|v>$ of an irrep Ω of $\mathcal{T}_r$ depends on the explicit state $|v>$ of Ω. Thus $|\phi_l>$ is not unique for an irrep Ω.

We will discuss how to deal with these issues later. For now, we note that unconfined excitations do not suffer from such complications, since the condensate takes on a constant value around unconfined excitations. Thus we expect the fusion rules of unconfined excitations to be associative, if we only consider their composition with other unconfined excitations. There should be a Hopf algebra $\mathcal{U}$, called the *unconfined symmetry algebra*, whose irreps are precisely the unconfined irreps, and whose fusion rules give the fusion channels of the unconfined excitations into other unconfined excitations.

To obtain $\mathcal{U}$, we first determine all unconfined irreps of $\mathcal{T}_r$. Then we take the intersection of the kernels of all unconfined irreps[11], and denote it by $\mathcal{I}$. Finally, we define the algebra

$$\mathcal{U} = \mathcal{T}_r / \mathcal{I} \tag{40}$$

This is an algebra because $\mathcal{I}$ is an ideal (i.e. an invariant subalgebra) of $\mathcal{T}_r$. Its irreps are precisely the unconfined irreps.

Our claim is that $\mathcal{U}$ *is a Hopf algebra*. Though we have not attempted to prove this in full generality, we found it to be true in all the cases we've worked out. In view of our discussion earlier, it is physically natural that $\mathcal{U}$ is a Hopf algebra, while $\mathcal{T}_r$ needn't be. However the mathematical proof if this conjecture may not be so easy, and this definitely deserves further study. Such a proof should make use of the trivial self braiding condition (32) that we formulated for the condensate $|\phi_0>$, because if this condition is dropped then we have found cases where $\mathcal{U}$ isn't a Hopf algebra.

3.5.3 Trivial braiding for $F(H_m/B) \otimes \mathbb{C}N$

In the next section, we will see that for the electric and defect condensates in a phase with $F(H_m) \times \mathbb{C}H_{el}$ symmetry, the residual symmetry algebra $\mathcal{T}_r$ takes on a special form:

$$\mathcal{T}_r = F(H_m/B) \times \mathbb{C}N, \tag{41}$$

where B is a subgroup of H_m, and N is a subgroup of H_{el} whose elements $n \in N$ satisfy

$$n \cdot B = B. \tag{42}$$

This last equation tells us that the action of $n \in N$ on H_m/B is well defined: $F(H_m) \times \mathbb{C}H_{el}$ is a modified quantum double, so that the action of $h \in H_{el}$ on H_m satisfies (A.29):

$$\forall g_1, g_2 \in H_m : h \cdot (g_1 g_2) = (h \cdot g_1)(h \cdot g_2). \tag{43}$$

Since $N \subset H_{el}$, all $n \in N$ also satisfy this equation. The action of n on H_m/B is given by:

$$n \cdot (gB) = (n \cdot g)(n \cdot B) = (n \cdot g)B. \tag{44}$$

$\mathcal{T}_r = F(H_m/B) \times \mathbb{C}N$ is a transformation group algebra, so we can use the canonical theorem on the irreps of transformation group algebras given in Appendix A. The irreps are labelled by an orbit T in H_m/B under the action of N, and an irrep τ of the normalizer N_T of a preferred element g_T of T. We denote irreps of $F(H_m/B) \otimes \mathbb{C}N$ by Ω_τ^T. The conditions (38) and (39) for Ω_τ^T to braid trivially with $|\phi_r>$ in Π_α^A reduce to

$$\sum_{h \in B} |\phi_r(\Gamma(h(x_\eta \cdot g_T^{-1}))x)> = \sum_{h \in B} |\phi_r(\Gamma(h)x)> \tag{45}$$

$$|\phi_r(x)>\neq 0, \Gamma(x\cdot g_A)\in N \Rightarrow \tau_{ij}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\zeta)1_{N_T}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\zeta)$$
$$= \tau_{ij}(x_\eta^{-1}x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta) \qquad (46)$$

where the x_η and x_ζ are chosen representatives of left N_T cosets of H.

The proof of these equations is rather lengthy. Some intermediate steps are

- Proof of (45)

$$R=\sum_{h,g\in G} P_g e\otimes P_h\Gamma(g)$$
$$P(P_h\Gamma(g))=1_N(\Gamma(g))\frac{1}{|B|}P_{hB}\Gamma(g)$$
$$\sum_{g,h}\varepsilon(P(P_g e))\Pi_\alpha^A(P_h\Gamma(g))|\phi_r(x)>=\sum_{g\in B}\frac{1}{|B|}|\phi_r(\Gamma(g)x)>$$
$$\Omega_\tau^T(P(P_h g))_{\eta,\zeta}^{i,j}=\frac{1}{|B|}1_N(\Gamma(g))\sum_{N_T}P_{hB}(x_\eta\cdot g_T)\delta_g(x_\eta n x_\zeta^{-1})\beta_{i,j}(n)$$
$$\sum_{h,g}\Omega_\tau^T(P(P_g e))_{\zeta,\eta}^{i,j}\Pi_\alpha^A(P_h\Gamma(g))|\phi_r(x)>$$
$$=\Omega_\tau^T(1)_{\zeta,\eta}^{i,j}\sum\varepsilon(P(P_g e))\Pi_\alpha^A(P_h\Gamma(g))|\phi_r(x)>$$
$$\iff \frac{1}{|B|}\tau_{i,j}(x_\eta^{-1}x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta)\sum_{h\in B}|\phi_r(\Gamma(hx_\eta\cdot g_T^{-1})x)>$$
$$=\frac{1}{|B|}\tau_{i,j}(x_\eta^{-1}x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta)\sum_{h\in B}|\phi_r(hx)>$$
$$\iff \sum_{h\in B}|\phi_r(\Gamma(h(x_\eta\cdot g_T^{-1}))x)>=\sum_{h\in B}|\phi_r(hx)>$$

- Proof of (46)

$$\sum_{h,g}\varepsilon(P(P_h\Gamma(g^{-1})))\Pi_\alpha^A(P_g e)|\phi_r(x)>=1_N(\Gamma(x\cdot g_A))|\phi_r(x)>$$
$$\sum_{h,g}\Omega_\tau^T(P(P_h\Gamma(g^{-1})))\Pi_\alpha^A(P_g e)\phi=$$
$$=\Omega_\tau^T(1)\sum_{h,g}\varepsilon(P(P_h\Gamma(g^{-1})))\Pi_\alpha^A(P_g e)|\phi_r(x)>$$
$$\iff 1_N(\Gamma(x\cdot g_A))\tau_{i,j}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\zeta)1_{N_T}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\zeta)|\phi_r(x)>$$
$$=1_N(\Gamma(x\cdot g_A))1_{N_T}(x_\eta^{-1}x_\zeta)\tau_{i,j}(x_\eta^{-1}x_\zeta)\phi(x)$$
$$\iff |\phi_r(x)>=0 \text{ or } \Gamma(x\cdot g_A^{-1})\notin N$$
$$\text{or } \tau_{i,j}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\zeta)1_{N_T}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\zeta)=1_{N_T}(x_\eta^{-1}x_\zeta)\tau_{i,j}(x_\eta^{-1}x_\zeta)$$

When the modified quantum double is a quantum double $D(H)$, the conditions for an irrep Ω_τ^T of $\mathcal{T}_r$ to braid trivially with the condensate $|\phi_r>$ become

$$\sum_{h\in B} |\phi_r(hx_\eta g_T^{-1} x_\eta^{-1} x) >= \sum_{h\in B} |\phi_r(hx)> \tag{47}$$

$$\begin{aligned} |\phi_r(x)> \neq 0, xg_A x^{-1} \in N \Rightarrow \tau_{ij}(x_\eta^{-1} x g_A^{-1} x^{-1} x_\zeta) 1_{N_T}(x_\eta^{-1} x g_A^{-1} x^{-1} x_\zeta) \\ = \tau_{ij}(x_\eta^{-1} x_\zeta) 1_{N_T}(x_\eta^{-1} x_\zeta). \end{aligned} \tag{48}$$

We will now use these equations to work out $\mathcal{T}_r$ and $\mathcal{U}$ for various condensates in modified quantum doubles. We start with electric condensates, and show that the conventional theory of electric condensates (Landau's theory) is reproduced. After that we study a variety of defect condensates.

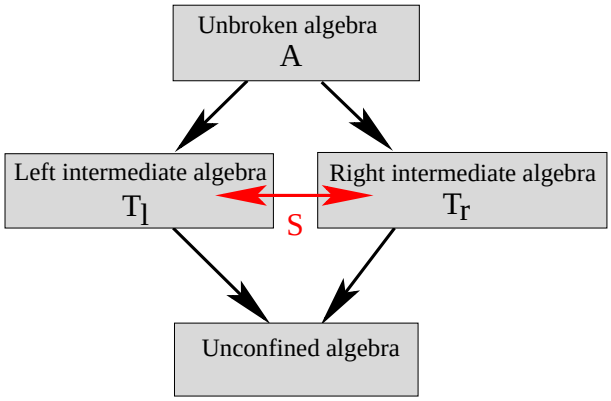

Fig. 5. The symmetry breaking scheme.

4 Condensates in modified quantum doubles

In the previous sections we formulated the general conditions which determine the residual symmetries in the case where a Hopf symmetry is spontaneously broken by the formation of a condensate in a state $|\phi_r>$. We were led to consider an intermediate algebra $\mathcal{T}_r$ (or $\mathcal{T}_l$) and an unconfined algebra $\mathcal{U}$ which would be manifest in the low energy spectrum of the broken phase. In this section we will present some rather general derivations of $\mathcal{T}_r$ and $\mathcal{U}$ for certain classes of vacua $|\phi_r>$.

We start with a brief discussion of "conventional" symmetry breaking by an ordinary (electric) type condensate. We recover the well known results with the added feature that the confinement of certain types of defect follows from

our formalism. This was already obtained in [2], but with a different definition of the residual symmetry algebra.

In the next subsection we analyze the situations that arise if one considers defect condensates. We point out that these can be of various types leading to different low energy phenomena. We refrain from discussing the case of mixed (or dyonic) condensates. These can definitely be studied using our formalism, but they have to be dealt with on a case to case basis, while for the case of ordinary and defect condensates we were able to extract general formulae.

4.1 Ordinary (electric) condensates

Consider a phase described by a quantum double $D(H)$, and condense a state $|\phi_r>$ of an electric irrep Π^e_α of $D(H)$. Then equation (21) tells us that the functions f in $\mathcal{T}_r$ satisfy

$$\begin{aligned} & f(x_1,y_1)\alpha(x^{-1}y_1) = f(x_1,y_1)\alpha(x^{-1}) \\ & f(x_1,y_1)\alpha(y_1) = f(x_1,y_1)\mathbf{1} \\ & f(x_1,y_1) = 0 \vee y_1 \in N_{\phi_r} \\ \Rightarrow\ & \mathcal{T}_r = F(H)\otimes \mathbb{C}N_{\phi_r} \end{aligned} \tag{49}$$

where N_{ϕ_r} is the stabilizer of $|\phi_r>$, i.e. the set of elements $h \in H$ that satisfy $\Pi^e_\alpha(h)|\phi_r> = \alpha(h)|\phi_r> = |\phi_r>$. $\mathbf{1}$ is the unit matrix. Since the condensate is purely electric, the magnetic group is unbroken, thus all of $F(H)$ is present in $\mathcal{T}_r$. $\mathcal{T}_r$ is a Hopf algebra in this case, which implies that the tensor product of irreps is well defined, and associative.

Some defects are confined. The trivial braiding conditions (47) and (48) tell us that only the defects $g \in N_{\phi_r}$ are unconfined. Thus the unconfined symmetry algebra is

$$\mathcal{U} = D(N_{\phi_r}). \tag{50}$$

One of the consequences of electric symmetry breaking is the lifting of degeneracies. Namely, excitations which used to be in the same irrep are now split into different irreps, which may have different energies. This splitting of energy levels is experimentally measurable, in principle.

To treat a phase with inversion symmetry, such as an achiral tetrahedral nematic with $F(\tilde{T})\times\mathbb{C}T_d$ symmetry (see [3]), we need the formulae for electric symmetry breaking of modified quantum doubles $F(H_m)\times\mathbb{C}H_{el}$. The derivation of $\mathcal{T}_r$ and $\mathcal{U}$ after the condensation of a state $|\phi_r>$ of an electric irrep

of a modified quantum double $F(H_m) \times \mathbb{C}H_{el}$ is analogous to the derivation given above for a quantum double $D(H)$. The result is

$$\mathcal{T}_r = F(H_m) \otimes \mathbb{C}N_{\phi_r} \tag{51}$$
$$\mathcal{U} = F(\Gamma^{-1}(N_{\phi_r})) \otimes \mathbb{C}N_{\phi_r} \tag{52}$$

For example, for $F(\tilde{T}) \otimes \mathbb{C}T_d$, $\Gamma^{-1}(N_{\phi_r}) = \tilde{N_{\phi_r}}$, so that $\mathcal{U} = F(\tilde{N_{\phi_r}}) \otimes \mathbb{C}N_{\phi_r}$.

As an example, we've worked out all possible electric breaking patters from $F(\tilde{T}) \otimes \mathbb{C}T_d$, see table 1. Although we haven't found references that systematically work out all electric condensates for all irreps as we have done, the theory behind electric condensates is well known and in that sense this is nothing new. Our interest was to rederive the results for ordinary condensates from the Hopf symmetry description of liquid crystals. A reference that offers a detailed analysis of the theory of ordinary condensates in bent-core nematic liquid crystals is [29], where a wealth of non-abelian phases are discussed. An analysis of the order of the phase transitions using Renormalization group calculations is done in [30].

As a side note, the transition from $F(\tilde{T}) \otimes \mathbb{C}T_d$ to $F(\tilde{D_2}) \otimes \mathbb{C}D_2$ in table 1, induced by the condensate of a state in the irrep E of T_d, is an example of spontaneous symmetry breaking from an achiral to a chiral phase, since D_2 does not contain any inversions or reflections, while T_d does. This may be the explanation of the experimental discovery of a phase built up of achiral molecules, whose symmetry is spontaneously broken to a chiral phase[32]. For a relevant discussion, see [29].

4.2 Defect (magnetic) condensates

There are different types of defect condensates which we wish to analyze. Consider a phase described by a quantum double $D(H)$, or a modified quantum double $F(H_m) \times \mathbb{C}H_{el}$, and pick a magnetic representation Π_1^A (1 is the trivial representation of the centralizer N_A). A basis of the vector space on which this irrep acts is given by $\{|g_i^A>\}$, where the g_i^A are the different defects in A. We consider the following types of condensates:

- Single defect condensate

$$|\phi_r> = |g_i^A> \tag{53}$$

- Class sum defect condensate

$$|\phi_r> = \sum_{g_i \in A} |g_i> =: |C_{g_A}> \tag{54}$$

Original symmetry	T_d Irrep of condensate	$\mathcal{T}_r$	$\mathcal{U}$
$F(\tilde{T}) \otimes \mathbb{C}T_d$	A_1	$F(\tilde{T}) \otimes \mathbb{C}T_d$	$F(\tilde{T}) \otimes \mathbb{C}T_d$
	A_2	$F(\tilde{T}) \otimes \mathbb{C}T$	$F(\tilde{T}) \otimes \mathbb{C}T$
	E	$F(\tilde{T}) \otimes \mathbb{C}D_2$	$F(\tilde{D_2}) \otimes \mathbb{C}D_2$
	E	$F(\tilde{T}) \otimes \mathbb{C}D_{2d}$	$F(\tilde{D_2}) \otimes \mathbb{C}D_{2d}$
	F_1	$F(\tilde{T}) \otimes \mathbb{C}S_4$	$F(\tilde{C_2}) \otimes \mathbb{C}S_4$
	F_1	$F(\tilde{T}) \otimes \mathbb{C}C_{1v}$	$F(\tilde{C_1}) \otimes \mathbb{C}C_{1v}$
	F_1	$F(\tilde{T}) \otimes \mathbb{C}C_3$	$F(\tilde{C_3}) \otimes \mathbb{C}C_3$
	F_2	$F(\tilde{T}) \otimes \mathbb{C}C_{1v}$	$F(\tilde{C_1}) \otimes \mathbb{C}C_{1v}$
	F_2	$F(\tilde{T}) \otimes \mathbb{C}C_{2v}$	$F(\tilde{C_2}) \otimes \mathbb{C}C_{2v}$
	F_2	$F(\tilde{T}) \otimes \mathbb{C}C_{3v}$	$F(\tilde{C_3}) \otimes \mathbb{C}C_{3v}$

Table 1
Electric condensates in a tetrahedral nematic. We use the notation for the irreps of T_d given in [31], and the standard crystallographic notation for groups (e.g. S_4 is not the permutation group of 4 elements, it is a rotary-reflection group).

where C_{g_A} is a conjugacy class in the $D(H)$ case, and an orbit in G under the action of H in the $F(H_m) \times \mathbb{C}H_{el}$ case. We denote the condensate by $|C_{g_A}>$, where g_A is the preferred element of A.

- Combined defect condensate

$$|\phi_r> = \sum_{g_i \in E} |g_i> \tag{55}$$

where E is a subset of the defects in one class. We need only take the elements to be within one class because, as we mentioned earlier, we need only study the cases where the condensate is the sum of vectors in the same irrep.

The single defect and class sum defect condensates are a special case of combined defect condensate. The derivation of $\mathcal{T}_r$ and $\mathcal{U}$ for a combined defect condensate is rather technical, so we will discuss the results for the single defect and class sum defect condensates first, and then derive the general formulae.

4.2.1 Single defect condensate

Consider a phase with $D(H)$ symmetry, and condense $|g_A>$ in the magnetic irrep Π_1^A. We're condensing the chosen preferred element in the conjugacy class A. This is not a restriction on our choice of defect, since g_A was chosen arbitrarily. The condensate $|g_A>$ satisfies the trivial self braiding condition (32).

The function $|\phi>: H \to \mathbb{C}$ that corresponds to the vector $|g_A>$ is (see appendix A)

$$|\phi(x)>= 1_{N_A}(x) \; \forall x \in H. \tag{56}$$

The criterion (21) that defines $\mathcal{T}_r$ becomes

$$\begin{aligned} & f \in \mathcal{T}_r \\ \iff \forall x, x_1, y_1 \in H : & \; f(x_1 x g_A x^{-1}, y_1) 1_{N_A}(y_1^{-1} x) = f(x_1, y_1) 1_{N_A}(x) \\ \forall x \notin N_A : & \; f(x_1, y_1) = 0 \quad \forall y_1 \notin N_A \\ \forall x \in N_A : & \; f(x_1 g_A, y_1) = f(x_1, y_1) \quad \forall y_1 \in N_A \\ \Rightarrow & \; \mathcal{T}_r = F(H/(g_A)) \otimes \mathbb{C}N_A \end{aligned} \tag{57}$$

where we define (g_A) to be the smallest subgroup of H that contains g_A.

This result for $\mathcal{T}_r$ has a very natural interpretation: the residual electric group is N_A, the subgroup of H that doesn't conjugate the defect. The magnetic part $H/(g_A)$ is not necessarily a group. It consists of left cosets of $(g_A) = \{\ldots g_A^{-1}, e, g_A, g_A^2, \ldots\}$. The defects are now defined modulo the condensate defect $|g_A>$. In other words, if a particle in a magnetic irrep of the residual symmetry $\mathcal{T}_r$ fuses with the condensate $|g_A>$, it is left unchanged. Thus its defect is defined modulo g_A.

Using our previous propositions, we can prove that $\mathcal{T}_r$ is a Hopf algebra $\iff$ (g_A) is a normal subgroup of H $\iff$ $H/(g_A)$ is a group.

The unconfined symmetry algebra is

$$\mathcal{U} = D(N_A/(g_A)). \tag{58}$$

If we condense another defect $|k g_A k^{-1}>$ in the conjugacy class A, the symmetry algebras are[12] :

$$\mathcal{T}_r = F(H/k(g_A)k^{-1}) \otimes \mathbb{C}k N_A k^{-1} \tag{59}$$

$$\mathcal{U} = D(N_A/k(g_A)k^{-1}). \tag{60}$$

The results for a single defect condensate $|g_A>$ in a phase with $F(H_m)\times\mathbb{C}H_{el}$ symmetry are analogous:

$$\mathcal{T}_r = F(G/(g_A))\otimes\mathbb{C}N_A \tag{61}$$
$$\mathcal{U} = F(N_A/(g_A))\otimes\mathbb{C}N_A/\Gamma((g_A)). \tag{62}$$

As an example, we've worked out all single defect condensates in an achiral tetrahedral nematic in table 2. For a discussion of the derivation of these results, and for single defect condensates in octahedral and icosahedral nematics, see [3].

Single defect condensate in $\mathcal{A}=F(\tilde{T})\otimes\mathbb{C}T_d$	K	$\mathcal{T}_r$	$\mathcal{U}$
$\|-e>$	$\tilde{C}_1$	$F(T)\otimes\mathbb{C}T_d$	$F(T)\otimes\mathbb{C}T_d$
$\|[123]>$	$\tilde{C}_3$	$F(T/C_3)\otimes\mathbb{C}C_3$	$D(e)$
$\|[(12)(34)]>$	$\tilde{C}_2$	$F(T/C_2)\otimes\mathbb{C}D_2$	$D(C_2)$
$\|-[(12)(34)]>$	$\tilde{C}_2$	$F(T/C_2)\otimes\mathbb{C}D_2$	$D(C_2)$

Table 2
Single defect condensates in a tetrahedral nematic.

4.2.2 Class sum defect condensates

Consider a phase with $D(H)$ symmetry, and condense the sum of the defects in the conjugacy class A:

$$|\phi_r>=\sum_{g_i^A\in A}|g_i^A>=:|C_{g_A}>\,. \tag{63}$$

A class sum defect condensate satisfies the trivial self braiding condition (32):

$$\begin{aligned}\mathcal{R}(|C_{g_A}>\otimes|C_{g_A}>) &= \mathcal{R}(\sum_{g_i^A\in A}|g_i^A>\otimes\sum_{g_k^A\in A}|g_k^A>)\\ &= \sum_{g_i^A\in A}(\sum_{g_k^A\in A}|g_i^A g_k^A(g_i^A)^{-1}>)\otimes|g_i^A>\end{aligned}$$

$$= \sum_{g_i^A \in A} (\sum_{g_k^A \in A} |g_k^A >) \otimes |g_i^A >$$
$$= |C_{g_A} > \otimes |C_{g_A} > .$$

In going from the second to the third line, we use the fact that $gAg^{-1} = A$ for any $g \in H$.

A class sum condensate doesn't break the electric group at all! Namely, conjugation acts trivially on a conjugacy class, since for any $g \in H$ we have

$$g \cdot |\phi_r >= g \cdot (\sum_{g_i^A \in A} |g_i^A >) = \sum_{g_i^A \in A} |g g_i^A g^{-1} >= \sum_{g_i^A \in A} |g_i^A >= |\phi_r > . \quad (64)$$

Thus this condensate is invariant under all residual symmetry transformations in H. For this reason, in the case of a local gauge theory this condensate is indeed the only physically admissible *gauge invariant magnetic condensate*. Namely, in the $D(H)$ phase of a gauge theory, with H a discrete group, the only residual gauge transformations are global, because H is discrete and the gauge transformation must be continuously defined on the space. These gauge transformations act trivially on the class sum defect condensates, thus these condensates are indeed gauge invariant.

The residual and unconfined symmetry algebras are

$$\mathcal{T}_r = F(H/K) \otimes \mathbb{C}H \quad (65)$$
$$\mathcal{U} = D(H/K), \quad (66)$$

where K is the smallest subgroup of H that contains the class A. From this definition, it follows that K is a normal subgroup of H. Thus H/K is a group.

If we condense a class sum defect condensate $|C_{g_A} >$ in a modified quantum double $F(H_m) \times \mathbb{C}H_{el}$, the outcome is

$$\mathcal{T}_r = F(H_m/K) \otimes \mathbb{C}H_{el} \quad (67)$$
$$\mathcal{U} = F(H_m/K) \otimes \mathbb{C}H_{el}/\Gamma(K). \quad (68)$$

As an example, we've derived all class sum defect condensates in an achiral tetrahedral nematic in table 3. The class sum defect condensates in an achiral octahedral, and an achiral icosahedral nematic are discussed in [3].

4.2.3 Combined defect condensates

The formal derivation

Defect conjugacy classes of $\mathcal{A} = F(\tilde{T}) \otimes \mathbb{C}T_d$	K	$\mathcal{T}_r$	$\mathcal{U}$
$\|C_{-e}>$	$\tilde{C}_1$	$F(T) \otimes \mathbb{C}T_d$	$F(T) \otimes \mathbb{C}T_d$
$\|C_{[123]}>, \|C_{-[123]}>, \|C_{[124]}>, \|C_{-[124]}>$	$\tilde{T}$	$\mathbb{C}T$	$D(e)$
$\|C_{[(12)(34)]}>$	$\tilde{D}_2$	$F(\mathbb{Z}_3) \otimes \mathbb{C}T_d$	$F(\mathbb{Z}_3) \otimes \mathbb{C}(\mathbb{Z}_3)_d$

Table 3
Class sum defect condensates in a tetrahedral nematic. $(\mathbb{Z}_3)_d$ is isomorphic to the permutation group of 3 elements.

We will now derive all the formulae for defect condensates we have come across.

Start with a phase with $F(H_m) \otimes \mathbb{C}H_{el}$ symmetry. Choose an irrep Π^A_α, and consider a condensate of the form $\sum_{g_i \in E} |g_i>$, with E a subset of the defects in one conjugacy class.

The demand of trivial self braiding (32) gives

$$\begin{aligned} & \mathcal{R}(\sum_{g_i \in E} |g_i> \otimes \sum_{g_k \in E} |g_k>) = \sum_{g_i \in E} |g_i> \otimes \sum_{g_k \in E} |g_k> \\ \Longleftrightarrow & \sum_{g_k \in E} \sum_{g_i \in E} |g_i g_k g_i^{-1}> \otimes |g_i> = \sum_{g_i \in E} \sum_{g_k \in E} |g_i> \otimes |g_k> \\ \Longleftrightarrow & \forall g_i \in E : \{g_i g_k g_i^{-1}\}_{g_k \in E} = \{g_k\}_{g_k \in E}. \end{aligned} \tag{69}$$

It would be interesting in itself to construct the general solution to this constraint, and to determine how many different defect condensates satisfy this criterion. For example the defect-antidefect condensate $|g> + |g^{-1}>$ always satisfies this criterion[13], as does the superposition any set of commuting elements in a certain conjugacy class, and class sum defect condensates. This trivial self braiding condition will play a crucial role in determining $\mathcal{T}_r$.

The derivation of $\mathcal{T}_r$ and $\mathcal{U}$ is rather formal. We give the results first, and then we derive them:

$$\begin{aligned} \mathcal{T}_r &= F(H_m/K) \otimes \mathbb{C}M_E \\ \mathcal{U} &= F(N_E/K) \otimes \mathbb{C}M_E/\Gamma(K). \end{aligned}$$

For the derivation, we must introduce various definitions. Define the following subset of H (which needn't be a subgroup):

$$V_E \subset H_{el} : \; V_E = \{x_i N_A\}_{g_i \in E} \tag{70}$$

where $N_A \subset H_{el}$ is the normalizer of the chosen preferred element g_A in A, and x_i satisfies $x_i g_A x_i^{-1} = g_i$. In function notation, the condensate wave function is

$$|\phi_r(x) >= 1_{V_E}(x) \quad \forall x \in H. \tag{71}$$

Define the following subgroup of H_{el}:

$$M_E \subset H_{el} : \; M_E = \{m \in H : \{m \cdot g_i\}_{g_i \in E} = \{g_i\}_{g_i \in E}\} \tag{72}$$

$$= \{m \in H_{el} : m V_E = V_E\}. \tag{73}$$

M_E is composed of the global symmetry transformations that leave the condensate invariant.

Also define

$$N_E \subset H_m : \; N_E = \{n \in H_m : \{n g_i n^{-1}\}_{g_i \in E} = \{g_i\}_{g_i \in E}\}. \tag{74}$$

Using (A.42): $\Gamma(g_1) \cdot g_2 = g_1 g_2 g_1^{-1} \; \forall g_1, g_2 \in H_m$, we can prove that

$$\Gamma^{-1}(M_E) = N_E \quad \text{and} \quad \Gamma(N_E) = M_E. \tag{75}$$

From this equation we can derive that the elements of M_E satisfy

$$|\phi_r(mx) >= |\phi_r(x) > \quad \forall m \in M_E, x \in H_{el}. \tag{76}$$

Finally, we need one more definition:

$$K \subset H_m : \; K = (\{g_i\}_{g_i \in E}), \tag{77}$$

where $(\{g_i\}_{g_i \in E})$ is the smallest subgroup of H_m that all the $g_i \in E$, i.e. the defects in the condensate.

The trivial self braiding equation (69) implies that $K \subset N_E$. Thus, according to (75) and (76)

$$\forall k \in K : \; \Gamma(k) \in M_E \quad \text{and} \quad \phi(\Gamma(k)x) = \phi(x). \tag{78}$$

The residual symmetry algebra $\mathcal{T}_r$ is given by the set of functions $f \in F(H_m \times H_{el})$ that satisfy (21):

$$f(x_1(x \cdot g_A), y_1) 1_{V_E}(y_1^{-1}x) = f(x_1, y_1) 1_{V_E}(x). \tag{79}$$

We will now prove that

$$\mathcal{T}_r = F(H_m/K) \otimes \mathbb{C}M_E. \tag{80}$$

Note that H_m/K need not be a group.

To prove (80), take $y_1 \notin M_E$. Then $\exists\ x \in V_E$ such that $y_1^{-1}x \notin V$. Namely, if such an x doesn't exist, then $y_1^{-1}V = V$, thus $y_1^{-1} \in M_E$ according to (73), and $y_1 \in M_E$.

Substitute an $y_1 \notin M_E$, and x with $y_1^{-1}x \notin V$, into (79). This gives

$$0 = f(x_1, y_1) \forall x_1 \in H_m \quad \text{if} \quad y_1 \notin M_E, \tag{81}$$

so that $\mathcal{T}_r \subset F(H_m) \otimes \mathbb{C}M_E$.

Now substitute $y_1 \in M_E$ into (79): $1_{V_E}(y_1^{-1}x) = 1_{V_E}(x)$ so the equation implies $f(x_1(x \cdot g_A), y_1) = f(x_1, y_1)$ for all $x_1 \in G, x \in V_E$. Acting with all the $x \in V_E$ on g_A gives us all the $g_i \in E$, thus $f(x_1 g_i, y_1) = f(x_1, y_1)$ for all $g_i \in E$. Thus, in the first component f must be constant on left K cosets, since K is generated by the g_i. Thus $\mathcal{T}_r = F(H_m/K) \otimes \mathbb{C}M_E$.

$\mathcal{U}$ is a little harder to extract. It is given by

$$\mathcal{U} = F(N_E/K) \otimes \mathbb{C}M_E/\Gamma(K) \tag{82}$$

For the case of a quantum double $D(H)$

$$\mathcal{U} = D(N_E/K). \tag{83}$$

To prove (82) and (83), we must find out which irreps Ω_τ^T of $F(H_m/K) \otimes \mathbb{C}M_E$ braid trivially with the condensate ϕ in the irrep Π_α^A of $F(H_m) \times \mathbb{C}H_{el}$.

Our residual symmetry algebra $\mathcal{T}_r$ is of the form (41), with $B = K$ and $N = M_E$. Thus we can use the conditions (45) and (46) to determine the irreps of $\mathcal{T}_r$ that braid trivially with the condensate. The unconfined symmetry algebra $\mathcal{U}$ is then the Hopf algebra whose irreps are precisely the unconfined irreps.

Equation (45) states that for an unconfined irrep Ω_τ^T, with g_T the preferred element in the orbit T:

$$\sum_{k\in K} |\phi_r(\Gamma(k(x_\eta \cdot g_T^{-1}))x) >= \sum_{k\in K} |\phi_r(\Gamma(k)x) > \quad \forall x \in H_{el}, \tag{84}$$

where the x_η are chosen representatives of left N_T cosets in M_E.

Using (78), equation (84) becomes

$$\begin{aligned}
&\sum_{k\in K} |\phi_r(\Gamma(x_\eta \cdot g_T^{-1})x) >= \sum_{k\in K} |\phi_r(x) > \ \forall x \in H_{el} \\
&\Rightarrow |\phi_r(\Gamma(x_\eta \cdot g_T^{-1})x) >= |\phi_r(x) > \ \forall x \in H_{el} \\
&\Rightarrow \Gamma(x_\eta \cdot g_T^{-1}) \in M_E \\
&\Rightarrow \Gamma(x_\eta \cdot g_T) \in M_E \\
&\Rightarrow x_\eta \cdot g_T \in N_E \quad \text{using (75).}
\end{aligned}$$

Choosing $x_\eta = e$, we get $g_T \in N_E$. (45) is actually equivalent to $g_T \in N_E$ in this case, because $x_\eta \cdot g_T \in N_E$ follows from $g_T \in N_E$. To prove this, note that $x_\eta \in M_E$, so $\{x_\eta \cdot g_i\}_{g_i\in E} = \{g_i\}_{g_i\in E}$. Thus

$$\begin{aligned}
&\{(x_\eta \cdot g_T)g_i(x_\eta \cdot g_T)^{-1}\}_{g_i\in E} = \{(x_\eta \cdot g_T)(x_\eta \cdot g_i)(x_\eta \cdot g_T)^{-1}\}_{g_i\in E} \\
&= \{(x_\eta \cdot (g_T g_i g_T)^{-1}\}_{g_i\in E} = \{x_\eta \cdot g_i\}_{g_i\in E} = \{g_i\}_{g_i\in E} \\
&\Rightarrow x_\eta \cdot g_T \in N_E
\end{aligned}$$

In proving the third equal sign we used the fact that $g_T \in N_E$.

Thus (45) has taught us that for an irrep Ω_τ^T to be unconfined, we must have $g_T \in N_E$. The magnetic part of the unconfined symmetry algebra $\mathcal{U}$ is therefore $F(N_E/K)$. From the definition of N_E and K, we can prove that

$$\forall n \in N_E : \ nKn^{-1} = K. \tag{85}$$

Thus K is a normal subgroup of N_E. N_E/K is the unconfined magnetic group.

Equation (46) further restricts Ω_τ^T:

$$\begin{aligned}
&|\phi_r(x) >\neq 0, \ \Gamma(x \cdot g_A) \in M_E \\
&\Rightarrow \tau_{ij}(x_\eta^{-1}\Gamma(x \cdot g_A^{-1})x_\zeta)1_{N_T}(x_\eta^{-1}\Gamma(x \cdot g_A^{-1})x_\zeta) = \tau_{ij}(x_\eta^{-1}x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta)
\end{aligned}$$

Choose an x such that $|\phi_r(x) >\neq 0$. This is equivalent to saying that $x \cdot g_A = g_k$ for some $g_k \in E$. Now choose $x_\eta = x_\zeta$ in (46):

$$\tau_{ij}(x_\eta^{-1}\Gamma(g_k^{-1})x_\eta)1_{N_T}(x_\eta^{-1}\Gamma(g_k^{-1})x_\eta) = \tau_{ij}(e) = \delta_{ij}. \tag{86}$$

Now $x_\eta^{-1}\Gamma(g_k^{-1})x_\eta = \Gamma(x_\eta^{-1}\cdot g_k^{-1}) = \Gamma((x_\eta^{-1}\cdot g_k)^{-1})$. Since $x_\eta \in M_E$ and $g_k \in K$, we have $(x_\eta^{-1}\cdot g_k) \in K$, so $(x_\eta^{-1}\cdot g_k)^{-1} \in K$. Thus $\Gamma((x_\eta^{-1}g_k)^{-1}) \in \Gamma(K)$. Now $\Gamma(K)$ acts trivially on the magnetic group N_E/K, due to (A.42). Thus necessarily $\Gamma(K) \subset N_T$, since the elements of $\Gamma(K)$ are normalizers of all elements of N_E, so they are also normalizers of g_T. This means that $1_{N_T}(x_\eta^{-1}\Gamma(g_k^{-1})x_\eta) = 1$. 86) becomes

$$\tau_{ij}(x_\eta^{-1}\Gamma(g_k^{-1})x_\eta) = \tau_{ij}(\Gamma(x_\eta^{-1}\cdot g_k^{-1})) = \tau_{ij}(\Gamma(x_\eta^{-1}\cdot g_k))^{-1} = \delta_{ij}.$$
$$\Rightarrow \tau_{ij}(\Gamma(x_\eta^{-1}\cdot g_k)) = \delta_{ij}.$$

Observe that the set $\{x_\eta^{-1}\cdot g_k\}_{\eta,k} = E$. Since τ must send all $\Gamma(x_\eta^{-1}\cdot g_k)$ to the unit matrix $\mathbf{1}$, τ must send all of $\Gamma(K)$ to the unit matrix (since K is generated by E). We conclude that $\Gamma(K)$ is in the kernel of τ, and the electric group is $M_E/\Gamma(K)$.

At the start of this last derivation, we filled in $x_\eta = x_\zeta$ in (46). The case $x_\eta \neq x_\zeta$ gives nothing new, because

$$1_{N_T}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\zeta) = 1_{N_T}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\eta x_\eta^{-1}x_\zeta) = 1_{N_T}(x_\eta^{-1}x_\zeta), \quad (87)$$

where in the last line we used a fact that we proved earlier: $x_\eta^{-1}\Gamma(x\cdot g_A)x_\eta \in N_T$. Thus (46) becomes

$$\begin{aligned} &\tau_{ij}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta) = \tau_{ij}(x_\eta^{-1}x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta) \\ \iff &\tau_{ij}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\eta)\tau_{ij}(x_\eta^{-1}x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta) = \tau_{ij}(x_\eta^{-1}x_\zeta)1_{N_T}(x_\eta^{-1}x_\zeta) \end{aligned}$$

This is equation is satisfied if $\tau_{ij}(x_\eta^{-1}\Gamma(x\cdot g_A^{-1})x_\eta)$, which we already proved.

Summarizing, the unconfined magnetic group is N_E/K, and the unconfined electric irreps are those that have $\Gamma(K)$ in their kernel, which means that the electric group is $M_E/\Gamma(K)$. Thus we have derived (82). Had we started with a quantum double $D(H)$ ($H = H_{el} = H_m$), the unconfined symmetry algebra becomes $\mathcal{U} = D(N_E/K)$, because in that case $M_E = N_E$.

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A Quasitriangular Hopf algebras

In this appendix we summarize the mathematical structure and notation connected to the Hopf algebras, which are extensively used in this paper and in previous literature. The applications focus on the quantum double $D(H)$ of a finite (possibly non-abelian) group H. The essential information is collected in a few comprehensive tables. In the text we merely comment on the meaning and the physical interpretation/relevance of the basic concepts. We conclude with a discussion of modified quantum doubles.

Algebras are an ubiquitous mathematical structure. Another - maybe less familiar mathematical notion - is that of a coalgebra, a structure that is in a precise sense a dual object to an algebra. A Hopf algebra is simultaneously an algebra and a coalgebra, with certain compatibility conditions between the two structures. In the following subsections we systematically go through some definitions and important examples.

A.1 Algebras, coalgebras, and their duals

An algebra $\mathcal{A}$ is a vector space over a field $\mathbb{F}$ (which we will take to be $\mathbb{C}$), with a bilinear multiplication. We can think of the multiplication as a map $\mu : \mathcal{A} \otimes \mathcal{A} \mapsto \mathcal{A}$. The algebras we discuss will all have a unit element, i.e. an element 1 which satisfies $1a = a1 = a \ \forall a \in \mathcal{A}$. Thinking of it as a map one could say that the unit embeds the field $\mathbb{F}$ into the center of the algebra. We write $\eta : \mathbb{F} \mapsto \mathcal{A}$. We require that η be an algebra morphism (the field is also an algebra, with itself as ground field). The algebras we consider are associative with respect to their multiplication.

An important example for our purposes is the *group algebra* of a finite group. Label the groups elements by h_i. The group algebra is then the set of objects of the form $\sum_i \lambda_i h_i$, with $\lambda_i \in \mathbb{C}$. This algebra is important, as it contains all the information about the group. For example, irreducible representations of the group algebra are in one-to-one correspondence with irreducible representations of the group.
In the physical models we consider, the regular (electric) modes transform under an irrep of the group, which is equivalent to saying that they transform under an irrep of the corresponding group algebra.

A coalgebra $\mathcal{C}$ is a vector space equipped with a comultiplication and counit. The comultiplication is a linear map $\Delta : \mathcal{C} \mapsto \mathcal{C} \otimes \mathcal{C}$. The counit is a linear map $\epsilon : \mathcal{C} \mapsto \mathbb{F}$. They must satisfy the formal relations:

Hopf algebra $\mathcal{A}$ e.g.: Group algebra $\mathbb{C}H$ Basis: $\{h_i\}$ $h_i \in H$			Dual Hopf algebra $\mathcal{A}^*$ Functions on the group $F(H)$ $\{f_i\}$ $f_i = f_{h_i} = P_{h_i}$ $f_i(x) = \delta_{h_i,x}$	
	Algebra		Dual algebra	
product	$\cdot$	$h_1 \cdot h_2 = h_1 h_2$	$\star$	$f_1 \star f_2(x) = (f_1 \otimes f_2) \circ \Delta(x)$
unit	e	$eh = he = h$	e^*	$e^*(x) = \varepsilon(x) = 1$
	Co-algebra		Dual co-algebra	
co-product	Δ	$\Delta(h_i) = h_i \otimes h_i$	Δ^*	$\Delta^*(f)\,(x,y) = f(x \cdot y)$
co-unit	ε	$\varepsilon(h) = 1$	ε^*	$\varepsilon^*\,(f) = f(e)$
antipode	S	$S(h) = h^{-1}$	S^*	$S^*(f)\,(x) = f(S(x)) = f(x^{-1})$

Table A.1
The defining relations of the group algebra and its dual, the algebra of functions on the group, as Hopf algebras.

$$\text{Coassociativity} : (\Delta \otimes id) \circ \Delta(c) = (id \otimes \Delta) \circ \Delta(c) \forall c \in \mathcal{C}$$
$$\text{Counit} : (\varepsilon \otimes id) \circ \Delta(c) = c = (id \otimes \varepsilon) \circ \Delta(c) \forall c \in \mathcal{C}$$

We see in Table 4 that the group algebra naturally has a coalgebra structure, where the comultiplication defines the action of the group on the tensor product space.

Given an algebra $\mathcal{A}$, its dual[14] $\mathcal{A}^*$ can be given a natural coalgebra structure. Similarly, given a finite dimensional coalgebra, its dual has a algebra structure (in a slightly less natural way reminiscent of the isomorphism $\mathcal{A}^{**} \simeq \mathcal{A}$ for a finite dimensional vector space). The structure of the duals are given in Table 4, and the structures are given explicitly for the group algebra and the function algebra, which are each other's duals. An example of a dual structure is the following: given a coalgebra $\mathcal{C}$ with a coproduct Δ, the dual $\mathcal{C}^*$ has the following multiplication (which we denote by a star $\star$): $f_1 \star f_2(x) \equiv (f_1 \star f_2) \circ \Delta(x)$. Thus the coproduct of $\mathcal{C}$ is used to define the multiplication in $\mathcal{C}^*$. Similarly, the counit of $\mathcal{C}$ is used to define the unit in $\mathcal{C}^*$.

$F(H)$, the set of functions from H to $\mathbb{C}$, is the coalgebra dual to the algebra $\mathbb{C}H$. It is also an algebra, and in fact it is a Hopf algebra (see next section). These functions form a vector space, which is spanned by the functions[15] $\{f_i = f_{h_i} : h_i \in H\}$ with $f_i(x) = \delta_{h_i}(x)$. A coalgebra structure is then defined by

$$\begin{aligned} &\varepsilon(\delta_{h_i}) = \delta_{h_i,e} \\ &\Delta(\delta_{h_i}) = \sum_{h_k \in H} \delta_{h_i h_k^{-1}} \otimes \delta_{h_k} \end{aligned} \tag{A.1}$$

We need only define these functions on a basis of $F(H)$ since they are linear. Using the definition of the coproduct one arrives upon evaluation on a pair of elements of $\mathcal{A}$ at,

$$\begin{aligned} &\Delta(f)(a_1, a_2) = f(a_1 a_2) \\ &\varepsilon(f) = f(1). \end{aligned} \tag{A.2}$$

A.2 Hopf algebra's

A Hopf algebra is simultaneously an algebra and a coalgebra, with certain compatibility conditions. Namely, we demand that ϵ and Δ be algebra morphisms. This is equivalent to demanding that η and μ be coalgebra morphisms (see [33]).
A Hopf algebra also contains an antipodal map S, which is the unique map from $\mathcal{A}$ to $\mathcal{A}$ that satisfies

$$\mu(S \otimes id)\Delta(a) = \mu(id \otimes S)\Delta(a) \tag{A.3}$$

From this definition, one can derive the following relations:

$$\forall a, b \in \mathcal{A} : S(ab) = S(b)S(a) \tag{A.4}$$
$$S(e) = e \tag{A.5}$$
$$(S \otimes S)\Delta = \Delta^{op} S \tag{A.6}$$
$$\varepsilon(S(h)) = \varepsilon(h) \tag{A.7}$$

Physically, the antipode map is used to construct antiparticle representations out of particle representations. This will become clear if we are to talk about representation theory, where given an irrep Π, we define the *antiparticle* or *conjugate* irrep as $\overline{\Pi}(a) = \Pi^t S(a)$, where t denotes transposition. $\overline{\Pi}$ is an irrep because S is an antimorphism, and so is the transposition. When we fuse

particle and antiparticle irreps, we don't necessarily get the vacuum representation, but it is guaranteed that ε is present in the fusion rules. This is completely analogous to the case of ordinary groups.

The standard example of a Hopf algebra is the *group algebra*. We have already discussed the algebraic and coalgebraic structure. It is turned into a Hopf algebra by defining the antipode S as the inverse:

$$S(g_i) = g_i^{-1} \tag{A.8}$$

So in fact all the familiar ingredients of a group(algebra) make it already into a Hopf algebra. Given a finite dimensional Hopf algebra $\mathcal{A}$, one can quite generally define a Hopf algebra structure on $\mathcal{A}^*$, and $\mathcal{A}^*$ is then called the dual Hopf algebra. The definitions are quite natural and are given in Table 4. Note that for the dual Hopf algebra $F(H) = \mathbb{C}H^*$ the antipodal map again involves the inverse, in the sense that $S^*(f)(x) = f(x^{-1})$.

In the context of our physical applications, the dual version of the algebra has very much to do with the topological defects and could be called the "magnetic" part of the algebra. When we have discussed the representation theory it will become clear what the physical states are and what the action of the elements of the dual algebra mean. One thing that may already be clear at this point is that, if the topological charge corresponds to a group element, then the comultiplication of the dual or function algebra determines the action on the tensor product representation of the dual, and therefore describes the fusion properties of the defects. It is then clear from the expressions (A.1) and (A.2) that the fusion indeed leads to the required multiplication of group elements.

A.3 Quasitriangularity

A Hopf algebra $\mathcal{A}$ is called quasitriangular when there is an invertible element R of $\mathcal{A} \otimes \mathcal{A}$ that satisfies:

$$(\Delta \otimes id)R = R_{13}R_{23}\,(id \otimes \Delta)R = R_{13}R_{12} \tag{A.9}$$

$$\forall a \in \mathcal{A}:\ R\Delta(a)R^{-1} = \Delta(a) \tag{A.10}$$

where if we write $R = \sum_{(R)} R^{(1)} \otimes R^{(2)}$, then

$$R_{ij} = \sum_{(R)} 1 \otimes \cdots \otimes R^{(1)} \otimes \cdots \otimes R^{(2)} \otimes \cdots \otimes 1$$

where $R^{(1)}$ is in the i-th, and $R^{(2)}$ in the j-th position.

Double algebra $\mathcal{D} = \mathcal{A}^* \otimes \mathcal{A}$ Ex: Hopf double algebra $D(H) = F(H) \otimes \mathbb{C}H$ Basis: $\{f_i \times h_j\}$ $f_i = P_{h_i} \in F(H), h_j \in H$		
	Algebra	
product	$\cdot$	$(f_1 \times h_1) \cdot (f_2 \times h_2)(x) = f_1(x) f_2(h_1 x h_1^{-1}) \times h_1 h_2$
unit	$\mathbf{e}$	$(1 \times e)(x) = e$
	Co-algebra	
co-product	Δ	$\Delta(f \times h)(x, y) = f(xy)h \otimes h$
co-unit	ε	$\varepsilon(f \times h)(x) = f(e)$
antipode	S	$S(f \times h)(x) = f(h^{-1}x^{-1}h)h^{-1}$
Central (ribbon) element	c	$c = \sum_h (f_h \times h)$
R-element $R \in \mathcal{D} \otimes \mathcal{D}$	R	$R = \sum_h (f_h \times e) \otimes (1 \times h)$

Table A.2
The defining relations of the quantum double $D(H)$ of a discrete group H.

The action of the R element on a tensorproduct of two representations is to "braid" the two particles and is of crucial importance to get the complete physical picture of particles and defects which exhibit nontrivial braid properties. From the first equations (A.9) one derives that the Yang-Baxter equation is satisfied. Equation (A.10) is the important statement that the generators of the braid group commute with the action of the Hopf algebra. This leads to the decomposition of multi-particle states in product representations of the braid and the Hopf algebra, thereby allowing for a clear definition of what we mean by quantum statistics and nonabelian anyons, etc. We will return to this subject shortly.

Given a Hopf algebra $\mathcal{A}$, there is a natural way to "double" it, creating a new Hopf algebra $D(\mathcal{A})$ called Drinfeld's quantum double of $\mathcal{A}$. As a vector space, $D(\mathcal{A}) = \mathcal{A}^* \otimes \mathcal{A}$, so it's a tensor product of $\mathcal{A}$ and its dual. For the discussion of the Hopf algebra structure on $D(\mathcal{A})$, see [33]. For our purposes, we need only know what the structure is like for H a discrete group. We also specify a braid matrix, making it a quasitriangular Hopf algebra.

As vector space, $D(H)$ is $F(H) \otimes \mathbb{C}H$, which is the same as $F(H \times H)$ for finite H. Denote the basis elements of $D(H)$ as $(f_i \times h_j)$ or $P_h g$ (the latter is the notation used in [6]), where $g \in H$, and $P_h \in F(H)$ is defined by $P_h(x) = \delta_{h,x}$ $(x \in H)$.

If we look upon $P_h g$ as an element of $F(H \times H)$, the function it corresponds to is $P_h g(x, y) = \delta_{h,x}\delta_{g,y}$. We can also define $g \equiv 1g = \sum_{h \in H} P_h g$, and $P_h \equiv P_h e$. We've therefore embedded H and $F(H)$ into $D(H)$, and from these embeddings obtained a basis of $D(H)$ (meaning that any element can be written as a sum of products of these basis elements). The structure of $D(H)$ is set by the following formulae:

$$P_h g P_{h'} g' = \delta_{h,gh'g^{-1}} P_h g g' \tag{A.11}$$

$$\Delta(P_h g) = \sum_{h' \in H} P_{hh'^{-1}} g \otimes P_{h'} g \tag{A.12}$$

$$\varepsilon(P_h g) = \delta_{h,e} \tag{A.13}$$

$$S(P_h g) = P_{g^{-1}h^{-1}g} g^{-1} \tag{A.14}$$

$$R = \sum_{g \in H} P_g e \otimes g \tag{A.15}$$

The complete structure is summarized in Table 5, including the ribbon element c (discussed later).

These relations can also be given in the language of functions in $F(H \times H)$. The notation that comes along with this formulation proves very convenient in actual calculations, so it is useful to give the corresponding definitions here:

$$(f_1 f_2)(x, y) = \sum_{h \in H} f_1(x, h) f_2(h^{-1}xh, h^{-1}y) \tag{A.16}$$

$$\Delta(f)(x_1, y_1; x_2, y_2) = f(x_1 x_2, y_1)\delta_{y_1}(y_2) \tag{A.17}$$

$$\varepsilon(f) = \sum_{z \in H} f(e, z) dz \tag{A.18}$$

$$S(f)(x, y) = f(y^{-1}x^{-1}y, y^{-1}) \tag{A.19}$$

$$R(x_1, y_1; x_2, y_2) = \delta_e(y_1)\delta_{x_1}(y_2) \tag{A.20}$$

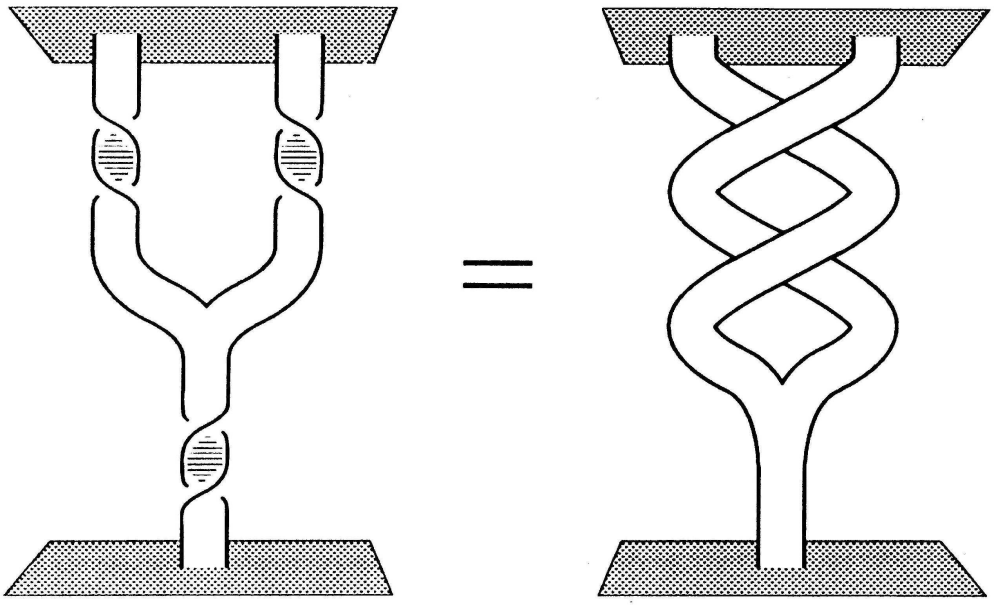

Fig. A.1. The suspender diagram gives the spin statistics relation in the case of nontrivial braid statistics. The equal sign denotes the topological equivalence between both pictures: unwinding the right hand side yields the left hand side. There is a direct relation with equation (A.22). The rotation of the "ribbons" is the result of acting with the central ribbon element c, while the braiding of the ribbons represents the action of R. The splitting represents the action of Δ .

This formulation is also important because it can be carried over to define the quantum doubles of continuous groups H, where H is compact, or locally compact. For results on such cases, see [34] and [35].

We finally should mention that the quantum double has a central element, denoted by c and called the ribbon element. Its eigenvalue can be used as a label on the representations of the $D(H)$. In applications it defines the phase obtained after rotating the particle state by 2π and is therefore called the *spin factor* of a given representation. It plays a central role in defining a suitable generalization of the spin statistics connection, to which we turn next.

A.5 A non-Abelian spin statistics connection

Let us briefly return to the important role played by the R element which as mentioned, generates the action of the braid group on a multi-particle states corresponding to some state in a tensor product of representations of $D(H)$. We noted already that the braiding with R commutes with action of $D(H)$:

$$\Delta^{op}(f \times h)R = R\Delta(f \times h) \tag{A.21}$$

where $\Delta^{op} \equiv \tau\Delta$ (i.e. Δ followed by a trivial permutation τ of the two strands). From this it follows that the n-particle states form representations of $D(H) \otimes B_n$, with B_n the braid group on n strands. We speak of non-abelian statistics if the theory realizes states that correspond to higher dimensional representations of B_n. In this context it is an important question to what extent one can still speak of a spin-statistics connection. One can indeed write down a generalized spin-statistics theorem in terms of the action of the braid group end the central ribbon element which corresponds to physically rotating the defect over an angle 2π and therefore generating the phase factor due to the (fractional) spin of the particle in question. It reads:

$$(c \otimes c)\Delta(a)c^{-1} = (R_{21}R_{12})\Delta(a) \; \forall a \in \mathcal{A}, \tag{A.22}$$

and can be represented graphically by the "suspenders diagram" depicted in Figure 6. Note that the concept of a spin statistics connection has become considerably more intricate. It is no longer an attribute carried by a single particle. It may involve two different particles (mutual statistics) and is also dependent on the channel in which they are fused.

A.6 Representation theory

The irreducible representations of $D(H)$ follow from the observation that it's a transformation group algebra (see [2] for details). Since we will come across other transformation group algebras in the main text, we give the general definition. As a vector space, the transformation group algebra is $F(X) \otimes \mathbb{C}H$, where X is a finite set and H is a finite group. A basis is given by

$$\{P_x h : x \in X, h \in H\}. \tag{A.23}$$

Just as in the quantum double case, we can consider it to be the vector space $F(X \times H)$ of functions

$$f : X \times H \to \mathbb{C}. \tag{A.24}$$

The element $P_x h \in F(X) \otimes \mathbb{C}H$ corresponds to the function

$$f(y, z) = \delta_{x,y}\delta_{h,z} \; \forall y \in X, z \in H. \tag{A.25}$$

Furthermore, there is an action of the group H on X. This means that the elements $h \in H$ act as bijections of X, in a manner that is consistent with the group structure (i.e. we have a homomorphism from H to bijections of X).

Denote the action of $h \in H$ on $x \in X$ by $h \cdot x$. We now turn this vector into an algebra, by introducing a multiplication.

Definition 6 *$F(X \times H)$ is a transformation group algebra if the multiplication of $P_x h$ and $P_{x'} h'$ is given by:*

$$P_x h P_{x'} h' = \delta_{x,h\cdot x'} P_x h h' \tag{A.26}$$

In function notation:

$$(f_1 f_2)(x,y) = \sum_{h \in H} f_1(x,h) f_2(h^{-1} \cdot x, h^{-1} y) \tag{A.27}$$

We define an *inner product* on $F(X \times H)$. We give it in both notations: in terms of elements $P_x h \in F(X) \otimes \mathbb{C}H$, and in terms of functions $f_i \in F(X \times H)$.

$$\begin{aligned} (P_x h, P_{x'} h') &= \delta_{x,x'} \delta_{h,h'} \\ (f_1, f_2) &= \sum_{x \in X, h \in H} f_1(x,h)^* f_2(x,h). \end{aligned} \tag{A.28}$$

We can split X up into orbits under the action of H. The orbit of an $x_0 \in X$ is given by $\{h \cdot x_0 : h \in H\}$. Call $\{A\}$ the collection of orbits. For each orbit A, choose a preferred element x_1^A, and define the *normalizer* N_A to be the subgroup of $h \in H$ that satisfy $h \cdot x_1^A = x_1^A$. The O_A and N_A play a central role in the determination of the irreps.

Theorem 7 *Choose an orbit A in X, a preferred element x_1^A of A, and an irrep α of N_A. The orbit $A = \{x_1^A, x_2^A, \ldots, x_n^A : x_i^A \in X\}$. Let h_i^A be any element in H such that $h_i^A \cdot x_1^A = x_i^A$. Then the h_i^A form representatives of left N_A cosets in H. Further call e_j the basis vectors of the vector space V_α on which the irrep α acts.*
An irreducible unitary representation Π_α^A of $F(X \times H)$ is given by inducing the irrep α. A basis of the vector space is $\{|x_i^A, e_j^\alpha >\}$, and the action of $P_x g \in F(X \times H)$ is given by

$$\Pi_\alpha^A(P_x g)|x_i^A, e_j^\alpha >= \delta_{x, g h_i^A \cdot x_1^A} |x_j^A, \alpha(n) e_j^\alpha >$$

where x_j^A is defined by $g h_i^A = h_j^A n, n \in N_A$. This is possible because $g h_i^A$ sits in a particular coset of H/N_A, and the h_i^A are representatives of the left N_A cosets.
Furthermore, all unitary irreducible representations are equivalent to some Π_α^A, and Π_α^A is equivalent to Π_β^B iff $O_A = O_B$ and α is equivalent to β.

Representations Π^A_α of $D(H) = F(H) \times \mathbb{C}H$		
representation	Π^A_α	$A \sim$ defect/magnetic label, $\alpha \sim$ ordinary/electric label
	A	$C_A \sim$ Conjugacy class (orbit of representative element h_A).
	α	$\alpha \sim$ is a representation of the normalizer N_A of h_A in H.
carrier space	V^A_α	$\|v>: H \to V_\alpha \ \{\|v(x)> \ \| \ \|v(xn)> = \alpha(n^{-1}) \ \|v(x)>, \ n \in N_A\}$
action of $D(H)$ on V^A_α		$\pi^A_\alpha(f \times h)\|v(x)> = f(xhx^{-1}) \ \|v(h^{-1}x)>$
central element	c	$\Pi^A_\alpha(c)\|v(x)> = \alpha(h_A^{-1})\|v(x)>$
spin factor	s^A_α	$s^A_\alpha \equiv \alpha(h_A^{-1})$
tensor products	$\Pi^A_\alpha \otimes \Pi^B_\beta$	$\Pi^A_\alpha \otimes \Pi^B_\beta(f \times h)V \otimes W \equiv \Pi^A_\alpha \otimes \Pi^B_\beta \Delta(f \times h)V \otimes W$
		Clebsch-Gordan series: $\Pi^A_\alpha \otimes \Pi^B_\beta = \sum_{C,\gamma} N^{AB\gamma}_{\alpha\beta C} \ \Pi^C_\gamma$

Table A.3
The defining properties for the representations Π^A_α of the quantum double $D(H)$ of a discrete group H.

Thus irreps of a transformation group algebra are labelled by an orbit A in X under the action of H, and an irrep α of the normalizer N_A of a chosen preferred element x^A_1 of A.

The notation in terms of basis elements makes the action of the irreps transparent. An alternate notation for the Hilbert space is $h^A_i \otimes |e^\alpha_j> \equiv |x^A_i, e^\alpha_j>$. Then the action of a global symmetry transformation $g \equiv 1g$ is simply

$$\Pi^A_\alpha(g)h^A_i \otimes |e^\alpha_j> = gh^A_i \otimes |e^\alpha_j> = h^A_j n \otimes |e^\alpha_j> = h^A_j \otimes |\alpha(n)e^\alpha_j> \equiv |x^A_i, \alpha(n)e^\alpha_j>$$

In words, the part of g that "shoots through" the defect acts on the electric part.

The function notation is rather opaque, but extremely useful in calculations in the main text. The Hilbert space of the irrep Π^A_α is given by:

$$F_\alpha(H, V_\alpha) = \{|\phi>: H \to V_\alpha || \phi(hn) >= \alpha(n^{-1})|\phi(h)>, \forall h \in H, \forall n \in N\}$$

To make contact with the notation above, $|\phi(h)>$ corresponds to the vector attached to the "flux" $h \cdot x^A_1$. For example, the function $|\phi>$ associated with $|x^A_i, e^\alpha_j>$ is $|\phi(h^A_i) >= 1_{h^A_i N_A}(h)|e^\alpha_j>$. To explain the rest of the definition, note that $\forall n \in N$

$$|x^A_i, e^\alpha_j > \equiv h^A_j \otimes |e^\alpha_j >= h^A_j n n^{-1} \otimes |e^\alpha_j >= h^A_j n \otimes |\alpha(n^{-1})e^\alpha_j >$$

which explains why $|\phi(xn) >= \alpha(n^{-1})|\phi(x)>$.
Then the action of $f \in F(X \times H)$ on $|\phi>$ under the irrep Π^A_α gives a new function $\Pi^A_\alpha(f)\phi \in F_\alpha(H, V_\alpha)$, defined by

$$(\Pi^A_\alpha(f)\phi)(h) = \sum_{z \in H} f(h \cdot x^A_1, z)|\phi(z^{-1}h)>$$

One easily checks that this is equivalent to the definition given above.

The quantum double $D(H)$ is a special case of a transformation group algebra, with $X = H$ and $h \cdot x = hxh^{-1}$. Thus in $D(H)$ the orbits of $X = H$ under the action of H are conjugacy class of $X = H$. N_A is the centralizer of the preferred element g_A of A.

We proved earlier on that the antipode can be used to create an antiparticle irrep from any irrep Π^A_α. In the case of $D(H)$, the antiparticle irrep of Π^A_α is $\Pi^{A^{-1}}_{\overline{\alpha}}$, where A^{-1} is the conjugacy class of g_A^{-1}, and $\overline{\alpha}(n) = \alpha^t(n^{-1})$. In particle, for an electric irrep Π^e_α the antiparticle irrep is $\Pi^e_{\overline{\alpha}}$. An example of this is the **3** irrep of $SU(3)$, which is the antiparticle irrep of **3**. The quarks transform under the **3** irrep, while the antiquarks transform under the $\overline{\mathbf{3}}$ irrep of $SU(3)$.

A.7 Modified quantum doubles

In the main text we came across phases whose Hopf symmetry was a variation on a quantum double, which we called a *modified quantum double*. We will now give the general definition of a modified quantum double[16].

As a vector space a modified quantum double is $F(H_m) \otimes \mathbb{C}H_{el}$, where H_m and H_{el} are two groups. It is also a transformation group algebra, so there is an action of H_{el} on H_m. We denote the action of $h \in H_{el}$ on $g \in H_m$ by $h \cdot g$. We require this action to satisfy the following relation:

$$\forall h \in H_{el},\ \forall g, g' \in H_m : h \cdot (gg') = (h \cdot g)(h \cdot g'). \tag{A.29}$$

A basis of $F(H_m) \times \mathbb{C}H_{el}$ is given by $\{P_g h : g \in H_m, h \in H_{el}\}$. $F(H_m) \times \mathbb{C}H_{el}$ is a Hopf algebra with the following structure:

$$\forall g, g' \in H_m, \forall h, h' \in H_{el} : P_g h P_{g'} h' = \delta_{g,h\cdot g'} P_g h h' \tag{A.30}$$

$$\Delta(P_g h) = \sum_{g' \in H_m} P_{g'} h \otimes P_{g'^{-1} g} h \tag{A.31}$$

$$\varepsilon(P_g h) = \delta_{g,e} \tag{A.32}$$

$$S(P_g h) = P_{h^{-1}\cdot g^{-1}} h^{-1} \tag{A.33}$$

These structures satisfy all the axioms of a Hopf algebra, thus $F(H_m) \times \mathbb{C}H_{el}$ is a Hopf algebra. Just as in the quantum double case, we can write elements of $F(H_m) \times \mathbb{C}H_{el}$ as functions in $F(H_m \times H_{el})$. An element of $F(H_m \times H_{el})$ is a function

$$f : H_m \times H_{el} \to \mathbb{C}. \tag{A.34}$$

The element $P_g h \in F(H_m) \times \mathbb{C}H_{el}$ corresponds to the function

$$f(y, z) = \delta_{g,y} \delta_{h,z} \; \forall y \in H_m, z \in H_{el}. \tag{A.35}$$

The structures of $F(H_m \times H_{el})$ are

$$\forall f, f_1, f_2 \in F(H_m \times H_{el}); x, x_1, x_2 \in H_m; y, y_1, y_2 \in H_{el} :$$

$$(f_1 \times f_2)(x, y) = \sum_{h \in H_{el}} f_1(x, h) f_2(h^{-1} \cdot x, h^{-1} y) \tag{A.36}$$

$$\Delta(f_1)(x_1, y_1; x_2, y_2) = f(x_1 x_2, y_1) \delta_{y_1}(y_2) \tag{A.37}$$

$$\varepsilon(f) = \sum_{h \in H_{el}} f(e, h) dh \tag{A.38}$$

$$S(f)(x, y) = f(y^{-1} \cdot x^{-1}, y^{-1}) \tag{A.39}$$

Making $F(H_m) \times \mathbb{C}H_{el}$ quasitriangular, i.e. introducing a braid matrix, is not that easy. We have found a way to do it that applies to all physical systems we looked at. We need a homomorphism:

$$\Gamma : H_m \to H_{el}, \tag{A.40}$$

that satisfies the following relations:

$$\forall g, g_1, g_2 \in H_m, h \in H_{el} : \tag{A.41}$$

$$\Gamma(g_1) \cdot g_2 = g_1 g_2 g_1^{-1} \tag{A.42}$$

$$\Gamma(h \cdot g) = h \Gamma(g) h^{-1}. \tag{A.43}$$

$F(H_m) \times \mathbb{C}H_{el}$ is then a quasitriangular Hopf algebra with the following braid matrix:

$$R = \sum_{g \in H_m} P_g e \otimes \Gamma(g). \tag{A.44}$$

The inverse of R is

$$R^{-1} = \sum_{g \in G} P_g e \otimes \Gamma(g^{-1}). \tag{A.45}$$

The quantum double $D(H)$ is a special case of a modified quantum double, with $H_m = H_{el} = H$, $h \cdot g = hgh^{-1}$ $\forall h, g \in H$, and $\Gamma \equiv id$.

B Defining a coproduct for $\mathcal{T}_r$

$\mathcal{T}_r$ is not always a Hopf algebra, because the coproduct of $\mathcal{A}$ does not necessarily satisfy $\Delta(\mathcal{T}_r) \subset \mathcal{T}_r \otimes \mathcal{T}_r$. It does satisfy $\Delta(\mathcal{T}_r) \subset \mathcal{A} \otimes \mathcal{T}_r$. This means that we cannot take the tensor product of irreps of $\mathcal{T}_r$, in other words we cannot fuse two particles in $\mathcal{T}_r$. We can only fuse particles in $\mathcal{T}_r$ with particles of $\mathcal{A}$ coming in from the left.

The reason we can't fuse particles of $\mathcal{T}_r$ is that some excitations of the condensate $|\phi_r>$ are confined. The condensate takes on the value $|\phi_r>$ to the right, and $|\phi_l> \neq |\phi_r>$ to the left of a confined excitation, in an irrep Ω of $\mathcal{T}_r$. Thus particles coming in from the left see the condensate $|\phi_l>$. Thus particles to the left of are excitations of $|\phi_l>$, and they transform under irreducible representations of the right residual symmetry algebra of $|\phi_l>$.

Matters are complicated further by the observation that given two states $|v_1>$ and $|v_2>$ in a confined irrep Ω, the value of the condensate to the left of $|v_1>$ need not be equal to the value the condensate takes to the left of $|v_2>$. Thus the condensate does not take a well defined value to the left of the irrep Ω.

One possible interpretation of a configuration with a confined excitation Ω, is that the condensate has only condensed to the right of Ω. To the left of Ω the system is in the unbroken phase, with $\mathcal{A}$ symmetry. Thus particles to the left of Ω are irreps of $\mathcal{A}$. We can fuse particles in irreps Π of $\mathcal{A}$ with Ω, and look at the fusion rules. The outcome of this fusion tells us what the particle in the irrep $\mathcal{A}$ can become when it enters the ordered phase.

This interpretation of a confined excitation is reasonable, but it is unsatisfactory. Namely, we still want to be a able to fuse the irreps of the residual

symmetry algebra. There should be a finite set of quantum numbers in the broken phase, and we should be able to tell which of these quantum numbers can form hadronic composites. We want to be able to talk about the fusion of the quantum numbers in the broken phase without bringing in the quantum numbers of $\mathcal{A}$. We know that if $\mathcal{T}_r$ is a Hopf algebra, we can fuse irreps of $\mathcal{T}_r$. We will now discuss how to fuse irreps of $\mathcal{T}_r$ when $\mathcal{T}_r$ is not a Hopf algebra.

B.1 A purely magnetic phase

Let us first study a simple system that only has magnetic degrees of freedom. Its symmetry algebra is $F(H)$, with H some finite group. A basis of H is given by $\{P_h : h \in H\}$, where P_h is a projection operator. It measures the flux of a configuration : if the flux of a state $|v>$ is h, then $P_h \cdot |v>= |v>$. If the flux of $|v>$ isn't h, then $P_h|v>= 0$.

Now consider a single defect condensate $|\phi_r>= |g>$. Then using (61) (with $N_A = \{e\}$) we find for the residual symmetry algebra:

$$\mathcal{T}_r = F(H/(g)), \tag{B.1}$$

where (g) is the subgroup of H generated by $|g>$. Let us assume that (g) is not a normal subgroup of H, so that $\mathcal{T}_r$ is not a Hopf algebra. A basis of $\mathcal{T}_r$ is given by $\{P_{h_i(g)} : h_i \in F\}$, where F consists of a set $\{h_i\}$ of representatives of the left (g) cosets in H. These projection operators measure defects modulo the condensed defect $|g>$. This is a consequence of our definition of operators in $\mathcal{T}_r$: the operators in $\mathcal{T}_r$ are the operators that do not notice when a particle fuses with the condensate $|g>$.

If we apply the coproduct Δ on a basis element $P_{h_k(g)}$ of $\mathcal{T}_r$, we obtain

$$\Delta(P_{h_k(g)}) = \sum_{h_i \in F} P_{h_k(g)h_i^{-1}} \otimes P_{h_i(g)} = \sum_{h_i \in F} P_{h_k h_i^{-1}(h_i g h_i^{-1})} \otimes P_{h_i(g)}. \tag{B.2}$$

This coproduct has a natural interpretation: if a projection operator $P_{h_i(g)}$ measures a defect on the right, then the projection operators on the left measure defects modulo $h_i(g)h_i^{-1}$. Thus $\phi_l = |h_i g h_i^{-1}>$: the condensate on the left is conjugated by h_i. Note that the defect of the particle that $P_{h_i(g)}$ measured is only defined modulo g, but this doesn't affect $|\phi_l>$, since $|h_i g^n g (h_i g^n)^{-1}>= |h_i g h_i^{-1}> \ \forall n \in \mathbb{Z}$.

We assume that there is a confined excitation $|v>$ in an Ω of $\mathcal{T}_r$, and that $P_{h_i(g)}|v>= |v>$. We have a condensate $|\phi_l>$ to the left of the confined excitation measured by $P_{h_i(g)}$ that is different from the condensate $|\phi_r>$ to

the right. We can, however, *redefine our projection operators to the left of the confined excitation.* The projection operators to the left of $|v>$ are defined modulo $|h_i g h_i^{-1}>$. A basis of these projection operators is $\{P_{h'_k(h_i g h_i^{-1})} : h'_k \in J\}$, where J consists of a set $\{h'_k\}$ of representatives of the left $(h_i g h_i^{-1})$ cosets in H.

To every projection operator $P_{h'_k(h_i g h_i^{-1})}$, we associate a projection operator $P'_{h_i(g),h'_k(g)}$ in $\mathcal{T}_r$:

$$P'_{h_i(g),h'_k(g)} = P_{h_i^{-1}h'_k(h_i g h_i^{-1})h_i} = P_{h_i^{-1}h'_k h_i(g)}. \tag{B.3}$$

Using this definition, we can define a coproduct Δ' for $\mathcal{T}_r$:

$$\Delta'(P_{h_k(g)}) = \sum_{h_i \in F} P'_{h_i(g),h'_k(g)} \otimes P_{h_i(g)} = \sum_{h_i \in F} P_{h_i^{-1}h'_k h_i(g)} \otimes P_{h_i(g)}. \tag{B.4}$$

Δ' is a map from $\mathcal{T}_r$ to $\mathcal{T}_r \otimes \mathcal{T}_r$. One can check that Δ' is an algebra morphism, i.e.

$$\Delta'(P_{h_j(g)} P_{h_k(g)}) = \Delta'(P_{h_j(g)})\Delta'(P_{h_k(g)}). \tag{B.5}$$

We can use Δ' to fuse two irreps Ω_2 and Ω_1 of $\mathcal{T}_r$:

$$\Omega_2 \otimes \Omega_1(a) = (\Omega_2 \otimes \Omega_1) \circ \Delta'(a). \tag{B.6}$$

Δ' is not coassociative, i.e.

$$(\Delta' \otimes id) \circ \Delta' \neq (id \otimes \Delta') \circ \Delta'. \tag{B.7}$$

This implies that the tensor product of three irreps Ω_3, Ω_2 and Ω_1 of $\mathcal{T}_r$ is not associative:

$$(\Omega_3 \otimes \Omega_2) \otimes \Omega_1 \neq \Omega_3 \otimes (\Omega_2 \otimes \Omega_1). \tag{B.8}$$

The interpretation of this non coassociativity is as follows: when we take the tensor product $\Omega_2 \otimes \Omega_1$, Ω_2 is defined *with respect to the condensate to the right of* Ω_1. If a third particle Ω_3 comes in from the left, then it becomes defined with respect to the condensate *between* Ω_2 *and* Ω_1. So we have a natural ordering for the tensor product, namely we must take $(\Omega_3 \otimes \Omega_2) \otimes \Omega_1$, which corresponds to having Ω_1 in the system, then bringing in Ω_2 from the left, and then bringing in Ω_3. $\Omega_3 \otimes (\Omega_2 \otimes \Omega_1)$ is unphysical, because it isn't clear how the particles were put in the system.

In general, if we bring in $\Omega_1, \Omega_2, \ldots, \Omega_n$ from the left in that order, the resulting configuration is $((\ldots(\Omega_n \otimes \Omega_{n-1}) \otimes \Omega_{n-2}) \otimes \ldots) \otimes \Omega_2) \otimes \Omega_1$. We have been forced to introduce an ordering in our fusion. This ordering corresponds to an ordering in the coproduct:

$$(\Delta' \otimes id \otimes \ldots \otimes id) \circ \ldots \circ (\Delta' \otimes id \otimes id) \circ (\Delta' \otimes id) \circ \Delta'(a). \tag{B.9}$$

Once we have defined a non coassociative coproduct Δ', we can fuse confined excitations $\mathcal{T}_r$, and study the possible hadronic composites. To define a Δ', we need a linear map γ with the following properties:

$$\begin{array}{l}
\gamma : \mathcal{A} \otimes \mathcal{T}_r \mapsto \mathcal{A} \otimes \mathcal{T}_r \\
\forall a, b, \in \mathcal{T}_r : \gamma(\Delta(ab)) = \gamma(\Delta(a))\gamma(\Delta(b)) \\
\gamma|_{\Delta(\mathcal{T}_r)} : \Delta(\mathcal{T}_r) \mapsto \mathcal{T}_r \otimes \mathcal{T}_r \text{ is injective} \\
\gamma(\Delta(a \ \ mod(\mathcal{I})) = \Delta(a) mod(\mathcal{I} \otimes \mathcal{I})
\end{array}$$

where $\mathcal{I}$ was a subideal of $\mathcal{T}_r$ such that $\mathcal{U} = \mathcal{T}_r/\mathcal{I}$. The last demand is equivalent to demanding that we do not alter Δ at the level of $\mathcal{U}$. Then we define

$$\Delta'(a) = \gamma \circ \Delta(a). \tag{B.10}$$

We can be more explicit in the case of a condensate in a phase described by a modified quantum double $F(H_m) \times \mathbb{C}H_{el}$, when the right residual symmetry algebra $\mathcal{T}_r$ is of the form (41) : $\mathcal{T}_r = F(H_m/B) \otimes N$. We've seen that $\mathcal{T}_r$ is of this form for electric condensates, and defect condensates. The action of N on B is well defined, because N acts trivially on B. For every orbit A_i in G/B under the action of N, pick a preferred element g_i. Now define a map $\sigma : H_m/B \mapsto H_m$ such that $\sigma(g_iB) \in g_iB$. In other words, σ sends every left B coset g_iB into a chosen element in g_iB. We demand that[17] for all $n \in N$

$$\sigma(n \cdot g_iB) = n \cdot \sigma(g_iB). \tag{B.11}$$

Using σ, we define

$$\begin{array}{l}
\gamma : (F(H_m) \times \mathbb{C}H_{el}) \otimes (F(H_m/B) \otimes N) \rightarrow \\
\quad \rightarrow (F(H_m) \times \mathbb{C}H_{el}) \otimes (F(H_m/B) \otimes N) \\
\gamma(P_g n \otimes P_{g'B} n') = P_{\sigma(g'B)^{-1} g \sigma(g'B)} n \otimes P_{g'B} n'.
\end{array} \tag{B.12}$$

It is straightforward but lengthy to verify that $\gamma|_{\Delta(\mathcal{T}_r)}$ is an algebra morphism[18]. Thus we have a coproduct

$$\Delta'(P_{gB}n) = \gamma \circ \Delta(P_{gB}n) = \sum_{g_i} P_{\sigma(g_iB)^{-1}gB}n \otimes P_{g_iB}n. \tag{B.13}$$

B.3 An example: $\mathcal{A} = D(\overline{D_{2n}})$

As an example, we consider a phase described by the quantum double of the double cover of an even dihedral group: $\mathcal{A} = D(\overline{D_{2n}})$. The group $\overline{D_{2n}}$ has the following structure:

$$\overline{D_{2n}} = \{r^k s^m : k = 0, 1, \ldots, 2n-1; m = 0, 1\} \tag{B.14}$$

with $r^{2n} = s^2 = -e$.

Now condense a single defect[19] $|s>$. The right residual and unconfined symmetry algebras are

$$\mathcal{T}_r = F(\overline{D_{2n}}/(s)) \otimes \mathbb{C}\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2} \tag{B.15}$$
$$\mathcal{U} = D(\mathbb{Z}_2), \tag{B.16}$$

where $\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2} = \{r^{kn}s^m : k = 0, 1, 2, 3; m = 0, 1\}$.

The irreps of $\mathcal{T}_r$ are given in Table 7, and the irreps of $\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2}$, which occurs as a centralizer of two orbits in $\overline{D_{2n}}/(s)$, are given in Table 8.

We write the basis of the irrep Ω^i_k as $|i>, |-i>$, where $|i>$ corresponds to the defect $|r^i(s)>$ (remember that the defects are defined modulo the condensate $< s >$). The action of $\mathcal{T}_r$ on this basis is set by

$$\begin{aligned} &P_{r^j(s)} \cdot |i> = \delta_{j,i} \\ &r^{tn} \cdot |i> = (-1)^{tk}|i> \quad t = 0, 1, 2, 3 \\ &s|i> = |-i> . \end{aligned}$$

We can write this compactly in one equation:

$$P_{r^j(s)} r^{tn} s^l |i> = \delta_{j,(1-2l)i}(-1)^{tk}|(1-2l)i> \quad t \in \mathbb{Z}_4, l \in \mathbb{Z}_2. \tag{B.17}$$

The left (s) cosets in $\overline{D_{2n}}/(s)$ are $r^i(s) = \{r^i, -r^i, r^i s, -r^i s : i = 0, 1, \ldots, 2n-1\}$. To define the coproduct, we must choose $\sigma(r^i(s)) \in r^i(s)$ for every i. If we choose $\sigma(r^i(s)) = r^i$, then the coproduct Δ' is coassociative. This can be

Preferred element g_A	e	r^n	r^i $i = 0, 1, \ldots, n-1$
Orbit A in $\overline{D_{2n}}/(s)$	$(s) = \{e, s, -e, -s\}$	$r^n(s)$	$\{r^i(s), r^{-i}(s)\}$
Normalizer N_A	$\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2}$= $\{r^{tn}s^k :$ $t \in \mathbb{Z}_4, k \in \mathbb{Z}_2\}$	$\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2}$= $\{r^{tn}s^k :$ $t \in \mathbb{Z}_4, k \in \mathbb{Z}_2\}$	$\overline{\mathbb{Z}_2}$= $\{e, s, -e, -s\}$
Irrep of N_A	$\alpha_{k,l}$ $k \in \mathbb{Z}_4, l \in \mathbb{Z}_2$	$\alpha_{k,l}$ $k \in \mathbb{Z}_4, l \in \mathbb{Z}_2$	β_m $m \in \mathbb{Z}_4$
Irrep of Ω	$\Omega^0_{k,l}$	$\Omega^n_{k,l}$	Ω^i_m
Unconfined irreps	$\Omega^0_{k,0}$ $k = 0, 2$	$\Omega^n_{k,0}$ $k = 0, 2$	

Table B.1
The irreps of $\mathcal{T}_r = F(\overline{D_{2n}}/(s)) \otimes \mathbb{C}\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2}$.

traced back to the fact that the r^i form a group. Note that in general, it is not possible to choose representatives of the cosets so that they form a group.

Using Δ', we can determine the following fusion rule:

$$\Omega^i_k \otimes \Omega^j_l = \Omega^{i+j}_{k+l} \oplus \Omega^{i-j}_{k+l} \tag{B.18}$$

Element of $\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2}$	e	$-e$	r^n	r^{-n}	s	$-s$	sr^n	sr^{-n}
Irrep $\alpha_{k,l}$ $k \in \mathbb{Z}_4,\ l \in \mathbb{Z}_2$	1	$(-1)^k$	i^k	i^{-k}	$(-1)^l i^k$	$(-1)^l i^{-k}$	$(-1)^{l+k}$	$(-1)^m$

Table B.2
The irreps of $\overline{\mathbb{Z}_2 \rtimes \mathbb{Z}_2}$.

The unconfined irreps are given in Table 8. This fusion rules implies, for example, that Ω_1^i and Ω_1^{n-i} can fuse to Ω_2^n, which is unconfined. Thus we have made a hadronic composite.

If we choose $\sigma(r^i(s)) = r^i s$, the fusion rules are the same, even though the coproduct is not coassociative. However, in this case the definition of the particles is altered. Namely, if an irrep Ω_k^i is present to the right, then $|j>$ in the irrep Ω_l^j coming in from the left should be interpreted as $|-j>$. This can be checked by applying the projection operators:

$$P_{r^l(s)} \cdot (|j> \otimes |i>) = \delta_{l,-j+i} |j> \otimes |i> \tag{B.19}$$

We have therefore discovered that our choice of coproduct Δ' alters the meaning of the labels of our irreps, when they are to the right of a confined excitation. The reason for this is the following: we can determine the irreps of $\mathcal{T}_r$, and that gives us a finite set of labels. When we have one particle in an irrep of $\mathcal{T}_r$ in the system, its meaning is unambiguous. Now if a confined excitation is present, then we know that the residual symmetry algebra of the condensate ϕ_l to the left of the excitation may be different from the right residual symmetry algebra $\mathcal{T}_r$. However, we still want to use the same labels for particles to the left of the confined excitations, because they are excitations of a symmetry algebra isomorphic to $\mathcal{T}_r$. Thus we are adding information to the labels of irreps of $\mathcal{T}_r$, namely we are defining their meaning when they appear to the left of a confined excitation. They should not be considered as the same particle: for example, $|i> |i>$ should not be interpreted as the fusion of $|i>$ with itself. The correct interpretation is that we have the confined excitation $|i>$ of $\mathcal{T}_r$, and we brought in a particle from the left, which under our choice of coproduct is labelled by $|i>$.

In summary, for the $|s>$ condensate in $D(\overline{D_{2n}})$, the fusion rules are independent of the choice of coproduct. We do not expect this to be a general result

(although in the cases we've worked out the fusion rules don't depend on the choice of coproduct). The *physics* of the phase, on the other hand, should not depend on our choice of coproduct, since this choice boils down to a definition of our labels. An interesting follow up on this research would be to study the influence of the choice of coproduct on the fusion rules.

Notes

1 : Modified quantum doubles are a slight variation on quantum doubles. Their precise definition is discussed in Appendix A.

2 : We assume here that G is simply connected. If it isn't, we can take the universal covering group $\tilde{G}$ of G, and then H should be

replaced by the lift $\tilde{H}$ of H in the universal covering group $\tilde{G}$ in all formulae.

3 : A comprehensive summary of the algebraic framework of Hopf algebras, focusing on our specific needs is given in Appendix A. For an extensive introduction to the use of Hopf algebras in the present physical context we refer to [6].

4 : See Appendix A, and for an extensive background [18].

5 : Actually, if we are braiding two indistinguishable electric particles, then the braiding may give a phase factor. Under half-braiding the wavefunction of the system picks up a phase factor $e^{i2\pi s}$, where s is the spin of the particles.

6 : The difference with standard Riemann geometry is that one doesn't assume that the metric is torsion free, i.e. the Christoffel symbols $\Gamma^{\kappa}_{\mu\nu}$ are not required to satisfy $\Gamma^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\nu\mu}$. In the Cartan formulation the spin connection and the vielbein are considered as independent, while in Riemannian geometry they are linked by the no torsion condition.

7 : In a gauge theory, this amounts to picking a gauge such that the Dirac string points 'upwards' in the drawings. In the global case, we orient the defect such that the frame dragging happens in the upper half plane.

8 : We thank dr. Joost Slingerland for discussions on this point. He studied this relation independently.

9 : $\mathcal{T}_r^{\perp}$ is the vector space consisting of all vectors in $\mathcal{A}$ perpendicular to all vectors in $\mathcal{T}_r$.

10 : We can find such a basis of $\mathcal{A}$ using the Gram-Schmidt orthogonalization procedure.

11 : The kernel of a map is the set of elements which the irrep maps to zero. These elements form an ideal I, meaning that if $i \in I$ and $a \in \mathcal{T}_r$, then $ia \in I$ and $ai \in I$.

12 : To prove this, use $< kg_Ak^{-1} >= k < g_A > k^{-1}$ and $N_{kg_Ak^{-1}} = kN_Ak^{-1}$.

13 : Note that g and g^{-1} needn't be in the same conjugacy class.

14 : The dual of an algebra $\mathcal{A}$ is defined as $\mathcal{A}^* = F(\mathcal{A})$, the set of linear functions from $\mathcal{A}$ to $\mathbb{C}$.

15 : Different, strictly equivalent notations are frequently used here: $f_i = f_{h_i} = P_{h_i}$. In particular we have the identities $f_i(x) = \delta_{h_i}(x) = \delta_{h_i x^{-1}}(e) = \delta_{h_i,x}$, the last delta being the usual Kronecker delta.

16 : We came up with this structure to deal with phases with inversion or reflection symmetries. It turns out to be a special case of a bicrossproduct of Hopf algebras, see e.g. [18].

17 : It is not at all clear that this can always be achieved, even though it is possible in the simple cases we've looked at.

18 : γ is not an algebra morphism of $D(H) \otimes \mathcal{T}_r$, it is only an algebra morphism when restricted to $\Delta(\mathcal{T}_r)$.

19 : We take even dihedral groups, because in $D(\overline{D_{2n+1}})$, condensing $|s>$ yields $\mathcal{U} = \mathbb{C}e$, which is slightly less interesting, but note that we can still study hadrons! Only the hadron must be in the trivial irrep, since that is the only unconfined irrep.

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